

10-COMMUTATOR AND 13-COMMUTATOR

A.S. DZHUMADIL'DAEV

ABSTRACT. Skew-symmetric sum of $N!$ compositions of N vector fields in all possible order is called N -commutator. We construct 10-commutator and 13-commutator on $Vect(3)$ and 10-commutator on a space of divergenceless vector fields $Vect_0(3)$. We show that 2-commutator, 10-commutator and 13-commutator form final list of N -commutators on $Vect(3)$ and under these polylinear operations $Vect(3)$ has a structure of sh-Lie algebra. We establish that the list of 2- and 10-commutators on $Vect_0(3)$ is also final. Constructions are based on calculations of powers of odd derivations.

Let (A, \circ) be an algebra with vector space A and multiplication \circ . Let $\mathbf{C} \langle t_1, \dots, t_k \rangle$ be a space of non-commutative non-associative polynomials. Any $f \in \mathbf{C} \langle t_1, \dots, t_k \rangle$ induces a k -ary map

$$f : \underbrace{A \times \dots \times A}_k \rightarrow A,$$

that corresponds to any $a_1, \dots, a_k \in A$ element $f(a_1, \dots, a_k)$ calculated in terms of multiplication \circ . If this map is trivial, i.e., $f(a_1, \dots, a_k) = 0$, for any $a_1, \dots, a_k \in A$ then $f = 0$ is said an *identity* on (A, \circ) . If f is polylinear, then f induces a k -ary multiplication on A . For example, if $s_2 = t_1 t_2 - t_2 t_1 \in \mathbf{C} \langle t_1, t_2 \rangle$, then

$$s_2(a, b) = a \circ b - b \circ a$$

is an ordinary commutator.

Let

$$s_k = \sum_{\sigma \in Sym_k} \text{sign } \sigma t_{\sigma(1)} (\dots (t_{\sigma(k-1)} t_{\sigma(k)}) \dots)$$

be standard skew-symmetric polynomial. Let $Diff_n$ be a space of differential operators with n variables. For simplicity assume that variables are from $\mathbf{C}[x_1, \dots, x_n]$. Let $Diff_n^{[d]}$ be a subspace of differential operators of order d :

$$Diff_n^{[d]} = \left\langle u \partial^\alpha \mid |\alpha| = \sum_{i=1}^n \alpha_i = d \right\rangle.$$

We can interpret differential operators of first order as vector fields and identify $Diff_n^{[1]}$ with a space of vector fields $Vect(n)$. Consider s_k as a k -ary operation on a space of differential operators $Diff_n$. So, $s_k(X_1, \dots, X_k)$ is a skew-symmetric sum of compositions of k operators $X_{\sigma(1)} \cdots X_{\sigma(k)}$ by all $k!$ permutations. In general, composition of k operators of orders d_1, \dots, d_k is a differential operator of order $d_1 + \dots + d_k$. Therefore,

$$X_1 \in Diff_n^{[d_1]}, \dots, X_k \in Diff_n^{[d_k]} \Rightarrow s_k(X_1, \dots, X_k) \in Diff_n^{[d_1 + \dots + d_k]}.$$

In fact differential order of $s_k(X_1, \dots, X_k)$ is less than $d_1 + \dots + d_k$. For example, differential order of $s_k(X_1, \dots, X_k)$ is no more than n , if all X_1, \dots, X_k are operators of order 1 (vector fields) on n -dimensional manifold for any k [2]. Moreover, for some k might happen that s_k will be well-defined operation on $Diff_n^{[1]}$:

$$X_1, \dots, X_k \in Diff_n^{[1]} \Rightarrow s_k(X_1, \dots, X_k) \in Diff_n^{[1]}.$$

In [1] is established that s_{n^2+2n-2} is well-defined on $Vect(n) = Diff_n^{[1]}$ and in [2] is proved that $s_{n^2+2n-1} = 0$ is identity on $Vect(n)$. For example, $Vect(2)$ has 6-commutator and skew-symmetric identity of degree 7. Space of Hamiltonian vector fields on 2-dimensional plane $Vect_0(2)$ has 5-commutator and skew-symmetric identity of degree 6.

Question 1. ($n > 1$). Is it true that $N = n^2 + 2n - 1$ is index of nilpotency for operator D , i.e., $D^{n^2+2n-1} = 0$, but $D^{n^2+2n-2} \neq 0$?

We think that coefficient at $\prod_i \eta_i \prod_{(i,j) \neq (n,n)} \partial_i \eta_j \prod_{i \neq n} \partial_i^2 \eta_i \partial_n$ of D^{n^2+2n-2} is non-zero. Computer calculations on Mathematica shows that this coefficient is equal 1, 2, 3600 for $n = 2, 3, 4$.

Question 2. ($n > 3$). Is it true that s_{n^2+2n-2} is a unique N -commutator well-defined on $Vect(n)$, for $N > 2$? In other words, is it true that

$$D^N \in Der \mathcal{L}_n, n > 3, \Rightarrow N = 2 \text{ or } n^2 + 2n - 2?$$

In our paper we prove that for $n = 3$ answer to this question is negative. According to our results $Vect(3)$ has 2-commutator, 10-commutator and 13-commutator and this list of N -commutators is complete. Notice that 13-commutator is connected with a skew-symmetric identity of degree 14, but 10-commutator has no such connection with skew-symmetric identity of degree 11: s_{11} even is not well-defined operation on $Vect(3)$.

Let us give some quantitative parameters about D^{10} and D^{13} . D^{10} has three escort invariants. They have types $(2, 7, 1)$, $(3, 5, 2)$ and $(3, 6, 0, 1)$. It has 489 terms of type $(2, 7, 1)$, 3093 terms of type $(3, 5, 2)$, 480 terms of type $(3, 6, 0, 1)$ and all together 4062 terms.

D^{13} has one escort invariant. It has type $(3, 8, 2)$ and has 261 terms. In other words, $s_{10}(X_1, \dots, X_{10})$ can be presented as a sum of 4062 10×10 -determinants of three types. Similarly, $s_{13}(X_1, \dots, X_{13})$ can be presented as a sum of 261 matrices of order 13×13 .

To see that D^{10} is well-defined on $Vect_0(3)$, we change all terms of D^{10} like $\partial^\alpha \eta_3 \partial_3$, $\alpha_3 > 0$, to $-\partial^{\alpha-\varepsilon_3+\varepsilon_1} \eta_1 \partial_1 - \partial^{\alpha-\varepsilon_3+\varepsilon_2} \eta_2 \partial_2$. We obtain element with 864 terms, among them 82 has type $(2, 7, 1)$, 76 has type $(3, 6, 0, 1)$ and 706 has type $(3, 5, 2)$.

It is easy to see that D^k is a sum of compositions of the form $D \star_1 (D \star_2 \dots (D \star_{k-1} D) \dots)$, where $(\star_1, \dots, \star_{k-1})$ is a sequence of two symbols \circ or \bullet such that there are no two consecutive \bullet and whole number of \bullet is no more than $|I|$. In particular we see that the differential order of D^k is no more than $\min(k + 1/2, |I|)$. This estimate is not strong. One can see that, for $n = 2, 3$ differential orders of D^k are given as follows

$n = 2$														
k	1	2	3	4	5	6	7							
$\partial \deg D^k$	1	1	2	2	2	1	$-\infty$							

$n = 3$														
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\partial \deg D^k$	1	1	2	2	3	3	3	3	3	1	2	2	1	$-\infty$

If $D \in Der_0 \mathcal{L}_n$, i.e., $Div D = 0$, then the growth of differential orders of D^k given as

$n = 2$														
k	1	2	3	4	5	6	7							
$\partial \deg D^k$	1	1	2	2	1	$-\infty$	$-\infty$							

$n = 3$														
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\partial \deg D^k$	1	1	2	2	3	3	3	3	3	1	2	$-\infty$	$-\infty$	$-\infty$

We pay attention to a drammatrical jumping of $\partial \deg D^k$ in $(n, k) = (3, 10)$. Here we see that D^{10} is a derivation or that the 10-commutator is defined correctly on $Vect(3)$. One can check that $Div D^{10} = 0$ and hence 10-commutator is a well defined commutator on $Vect_0(3)$ also.

Theorem 0.1. *Let $D = \sum_{i=1}^3 \eta_i \partial_i \in Der \mathcal{L}_3$ be odd derivation. Then*

$$D^{10} \in Der \mathcal{L}_3 \otimes \mathbf{C}[x_1, x_2, x_3],$$

$$D^{13} \in Der \mathcal{L}_3 \otimes \mathbf{C}[x_1, x_2, x_3],$$

$$D^{14} = 0.$$

If $D^N \in \text{Der } \mathcal{L}_3 \otimes \mathbf{C}[x_1, x_2, x_3]$, then $N = 2, 10, 13$.

Theorem 0.2. Let $D = \sum_{i=1}^3 \eta_i \partial_i \in \text{Der } \mathcal{L}_3$ be odd derivation and $\text{Div } D = \sum_{i=1}^3 \partial_i \eta_i = 0$. Then

$$D^{10} \in \text{Der } \mathcal{L}_3 \otimes \mathbf{C}[x_1, x_2, x_3],$$

$$\text{Div } D^{10} = 0,$$

$$D^{11} = 0.$$

If $D^N \in \text{Der } \mathcal{L}_3 \otimes \mathbf{C}[x_1, x_2, x_3]$, and $\text{Div } D = 0$, then $N = 2$ or 10 .

1. sl_n -MODULE STRUCTURE ON $U = \mathbf{C}[x_1, \dots, x_n]$ AND $\text{Diff}_n(U)$

Endow U by a structure of module over Lie algebra $gl_n = \langle x_i \partial_j : i, j = 1, \dots, n, i \neq j \rangle$. Define an action of gl_n on generators of U by

$$x_i \partial_j (\partial^\alpha(u_s)) = -\delta_{i,s} \partial^\alpha(u_j) + \sum_{i=1}^n \alpha_j \partial^{\alpha - \epsilon_j + \epsilon_i}(u_s)$$

and prolong this action to U as an even derivation:

$$a(XY) = a(X)Y + Xa(Y),$$

for any $X, Y \in U$. Prolong the gl_n -module structure by natural way to $\text{Diff}_n(U)$. Notice that gl_n acts on $\text{Diff}_n(U)$ as a derivation

$$a(FG) = a(F)G + Fa(G),$$

and as gl_n -module subspaces $\langle u_i \partial_j : i, j = 1, \dots, n \rangle \subset \text{Diff}_n(U)$ and $\langle \partial_i(u_j) : i, j = 1, \dots, n \rangle$ are isomorphic to adjoint module.

Denote by π_1, \dots, π_{n-1} fundamental weights of sl_n and by $R(\gamma)$ the irreducible sl_n -module with highest weight γ . Let

$$\mathcal{D}^{[s]} = \langle \partial^\alpha : \alpha \in \mathbf{Z}_+^n, |\alpha| = s \rangle$$

and

$$U_s = \langle \partial^\alpha u_i : \alpha \in \mathbf{Z}_+^n, |\alpha| = s, i = 1, \dots, n \rangle.$$

Since ∂_i are even and u_i are odd elements, take place the following isomorphisms of sl_n -modules

$$\mathcal{D}^{[s]} \cong R(s\pi_1),$$

$$U_s \cong R(s\pi_1) \cong R(\pi_{n-1}).$$

In particular,

$$\mathcal{D} := \langle \partial_i : i = 1, \dots, n \rangle \cong R(\pi_{n-1}),$$

$$\mathcal{D}^{[2]} := \langle \partial_i \partial_j : i, j = 1, \dots, n \rangle \cong R(2\pi_1),$$

$$U_0 = \langle u_i : i = 1, \dots, n \rangle \cong R(\pi_1),$$

$$U_1 = \langle \partial_i(u_j) : i, j = 1, \dots, n \rangle \cong R(\pi_1) \otimes R(\pi_{n-1}) R(\pi_1 + \pi_{n-1}) \oplus R(0),$$

$$U_2 = \langle \partial_i \partial_j(u_s) : i, j, s = 1, \dots, n \rangle \cong R(2\pi_1) \otimes R(\pi_{n-1}) \cong R(2\pi_1 + \pi_{n-1}) \oplus R(\pi_1).$$

We use the following well-known isomorphisms without special mentioning:

$$\wedge^{n-1} R(\pi_1) \cong R(\pi_{n-1}),$$

$$\wedge^n R(\pi_1) \cong R(0),$$

$$\wedge^{n^2-1} R(\pi_1 + \pi_{n-1}) \cong R(\pi_1 + \pi_{n-1}),$$

$$\wedge^{n^2} R(\pi_1 + \pi_{n-1}) \cong R(0).$$

Lemma 1.1. $a(D^k) = 0$ for any $a \in gl_n$.

Proof. If $k = 1$ then action of $a \in gl_n$ corresponds to adjoint derivation and D corresponds to Euler operator. Therefore,

$$a(D) = [a, \sum_{i=1}^n u_i \partial_i] = 0.$$

If our statement is true for $k - 1$ then

$$a(D^k) = k D^{k-1} [a, D] = 0.$$

□

2. ESCORT INVARIANTS OF N -COMMUTATORS

Let $L = W_n$ be Witt algebra and $U = \mathbf{C}[x_1, \dots, x_n]$ be natural L -module. Then

- $L = \oplus_{i \geq -1} L_i$ is a graded Lie algebra,

$$L_s = \langle x^\alpha \partial_j : |\alpha| = s + 1 \rangle,$$

- $U = \oplus_{i \geq 0} U_i$ be associative commutative graded algebra with 1,

$$U_s = \langle x^\alpha : |\alpha| = s \rangle,$$

- L acts on U as a derivation algebra, i.e.,

$$X(uv) = X(u)v + u(Xv),$$

for any $X \in L, u, v \in U$ and

- this action is graded:

$$L_i U_j \subseteq U_{i+j}, \quad i \geq -1, j \geq 0.$$

In particular, L_0 is a Lie algebra isomorphic to gl_n and all homogeneous components L_s and U_s have structures of gl_n -modules. Then as sl_n -modules,

$$L_{-1} = \langle \partial_i : i = 1, \dots, n \rangle \cong R(\pi_{n-1}),$$

$$L_0 = \langle x_i \partial_j : i, j = 1, \dots, n \rangle \cong R(\pi_1) \oplus R(\pi_{n-1}) \oplus R(0),$$

$$L_1 = \langle x_i x_j \partial_s : i, j, s = 1, \dots, n \rangle \cong R(2\pi_1 + \pi_{n-1}) \oplus R(\pi_1).$$

Let M be graded L -module. It is called (L, U) -module if it has additional structure of graded module over U such that

$$X(um) = X(u)m + uX(m),$$

for any $X \in L, u \in U, m \in M$. Call M (L, U) -module with a base N if $N = M^{L_{-1}} = \langle m \in M : X(m) = 0, \forall X \in L_{-1} \rangle$ and M is free U -module with base N . If M_1, \dots, M_k and M are (L, U) -modules with bases and N is a base of M , then a space of polylinear maps $C(M_1, \dots, M_k; M) = \langle \psi : M_1 \times \dots \times M_k \rightarrow M \rangle$ is (L, U) -module with base and this base as a vector space is isomorphic to $C(M_1, \dots, M_k; M)$. In particular, to any L_{-1} -invariant polylinear map $\psi \in C(M_1, \dots, M_k; M)$ one can correspond some polylinear map $esc(\psi) \in C(M_1, \dots, M_k; N)$ called *escort* of ψ , by

$$esc(\psi)(m_1, \dots, m_k) = pr(\psi(m_1, \dots, m_k)),$$

where

$$pr : M \rightarrow N,$$

is a projection map to N , i.e., $pr(x^\alpha m) = \delta_{\alpha,0} m$. Inversly, for any $\phi \in C(M_1, \dots, M_k; N)$ one can correspond some L_{-1} -invariant polylinear map $\psi = E\phi \in C(M_1, \dots, M_k; M)$ by

$$E\phi(X_1, \dots, X_k) = \sum_{a_1 \in M_1, \dots, a_k \in M_k} E_{a_1}(X_1) \cdots E_{a_k}(X_k) \phi(a_1, \dots, a_k),$$

where a_i run basic elements of M_i of the form $x^\alpha n_i$, n_i run basic elements of a base of M_i . If $M_1 = \dots = M_k = L$ all are adjoint modules then

$$E_{x^\alpha \partial_i}(v \partial_j) = \delta_{i,j} \frac{\partial^\alpha(v)}{\alpha!}.$$

Details of such constructions see [3].

Apply this theory for L -module of differential operators $M = Diff_n = \langle u \partial^\alpha : u \in U, \alpha \in \mathbf{Z}_+^n \rangle$. Endow $M = Diff_n$ by grading:

$$M_s = \langle x^\alpha \partial^\beta : |\alpha| = s \rangle.$$

$Diff_n$ has a structure of associative algebra, in particular, it is a Lie algebra under commutator. As a Lie algebra it has a subalgebra isomorphic to W_n , and hence it has a structure of adjoint module over

W_n . Make $Diff_n$ U -module under action $u(v\partial^\beta) = uv\partial^\beta$. We see that $M^{L-1} = \langle \partial^\alpha : \alpha \in \mathbf{Z}_+^n \rangle$ and M is free U -module with base M^{L-1} . Therefore, $Diff_n$ is (L, U) -module.

Define $s_k \in C^k(L, M)$ by

$$s_k(X_1, \dots, X_k) = \sum_{\sigma \in \text{Sym}_k} \text{sign } \sigma X_{\sigma(1)} \cdots X_{\sigma(k)}.$$

We see that s_k is L_{-1} -invariant and graded:

$$\partial_i(s_k(X_1, \dots, X_k)) = \sum_{s=1}^k s_k(X_1, \dots, X_{s-1}, [\partial_i, X_s], X_{s+1}, \dots, X_k),$$

for any $\partial_i \in L_{-1}, X_1, \dots, X_k \in L$, and

$$s_k(L_{i_1}, \dots, L_{i_k}) \subseteq M_{i_1+\dots+i_k},$$

Fix some ordering on the set of basic elements of W_n . Let us take, for example, the following ordering: $x^\alpha \partial_i < X^\beta \partial_j$, if $i < j$ or $|\alpha| < |\beta|$ if $i = j$ or $\alpha < \beta$ in lexicographic order if $i = j$ and $|\alpha| = |\beta|$. As we mentioned above any L_{-1} -invariant cochain $C^k(W_n, Diff_n)$ can be restored by its escort. In particular, s_k can be restored by its escort. Any escort is defined as a polylinear map on its support. Call a subspace of k -chains $a_1 \wedge \cdots \wedge a_k \in \wedge^k L$ generated by basic vectors a_1, \dots, a_k such that

$$s_k(a_1, \dots, a_k) \in \langle \partial^\alpha : \alpha \in \mathbf{Z}_+^n \rangle$$

as a *support* of s_k . Then $\text{supp}(s_k)$ has a structure of sl_n -module as a sl_n -submodule of $\wedge^k L$. We know that sl_n -module $Diff_n^{L-1} = \langle \partial^\alpha : \alpha \in \mathbf{Z}_+^n \rangle$ is isomorphic to a direct sum of sl_n -modules $R(p\pi_{n-1})$:

$$\langle \partial^\alpha : |\alpha| = p, \alpha \in \mathbf{Z}_+^n \rangle \cong R(p\pi_{n-1}).$$

Then $\text{supp}(s_k)$ is also a direct sum of sl_n -submodules $\text{supp}_p(s_k)$, where $\text{supp}_p(s_k)$ is a sl_n -submodule of $\wedge^k L$ generated by support k -chains $a_1 \wedge \cdots \wedge a_k$ such that

$$s_k(a_1, \dots, a_k) \in \langle \partial^\alpha : \alpha \in \mathbf{Z}_+^n, |\alpha| = p \rangle.$$

So, we see that any standard skew-symmetric polynomial s_k induces a serie of sl_n -invariant maps

$$\text{supp}_p(s_k) \rightarrow R(p\pi_{n-1}).$$

Call such maps *escort invariants*. So, the calculation problem of k -commutators is equivalent to the problem of finding escort invariants.

Example. $\text{esc}(s_k) = 0$ if $k \geq n^2 + 2n - 1$ and s_{n^2+2n-2} has exactly one escort invariant $R(\pi_1) \otimes R(\pi_{n-1}) \otimes \wedge^{n-1} R(2\pi_1 + \pi_{n-1}) \rightarrow R(\pi_{n-1})$.

3. DIFFERENTIAL POLYNOMIALS SUPER-AGEBRA \mathcal{L}_n

Let \mathbf{Z} be set of integers, \mathbf{Z}_+ a set of non-negative integers, \mathbf{Z}^n a set of n -tuples $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbf{Z}$, $i \in I$, and $\mathbf{Z}_+^n = \{\alpha \in \mathbf{Z}^n | \alpha_i \geq 0, i \in I\}$. Let $\varepsilon_i \in \mathbf{Z}^n$ with i -th component 1 and other components are 0. Then $\alpha = \sum_{i=1}^n \alpha_i \varepsilon_i$, for any $\alpha \in \mathbf{Z}^n$. Set

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$

Endow sets \mathbf{Z}_+^n and $\mathbf{Z}_+^n \times \{1, \dots, n\}$ by linear ordering: $\alpha < \beta$, if

$$|\alpha| < |\beta|$$

or

$$|\alpha| = |\beta|, \alpha_1 = \beta_1, \dots, \alpha_{s-1} = \beta_{s-1}, \alpha_s > \beta_s,$$

for some $s = 1, \dots, n$. Set $(\alpha, i) < (\beta, j)$, if $i < j$ or $i = j, \alpha < \beta$.

Let \mathcal{L}_n be an super-commutative associative algebra over a field K generated by odd elements $e_{\alpha, i}$, where $\alpha \in \mathbf{Z}_+^n, i \in I$. Then

$$e_{\alpha, i} e_{\beta, j} = -e_{\beta, j} e_{\alpha, i},$$

$$e_{\alpha, i} (e_{\beta, j} e_{\gamma, s}) = (e_{\alpha, i} e_{\beta, j}) e_{\gamma, s},$$

for any $\alpha, \beta, \gamma \in \mathbf{Z}_+^n, i, j, s \in I$. Elements $e_{\alpha, i} e_{\beta, j} \dots e_{\gamma, s}$ with $(\alpha, i) < (\beta, j) < \dots < (\gamma, s)$ form base of \mathcal{L}_n . We fix this base and call such elements *base elements* of \mathcal{L}_n . Call number of indexes i, j, \dots, s of base element e as its *length* and denote $l(e)$.

Any base element of \mathcal{L}_n can be presented as $e = e^{[-1]} e^{[0]} e^{[1]} \dots e^{[r]}$, where $e^{[s]}$ is a product of ordered generators of a form $e_{\alpha, i}$ with $|\alpha| = s+1$. Call $e^{[s]}$ *s-component* of e and its length $l(e^{[s]})$, denote it $l_s(e)$, call as *s-Length* of e . Thus,

$$l(e) = \sum_{i \geq -1} l_i(e).$$

Let $\partial_i = \frac{\partial}{\partial_i}, i \in I$, are partial derivations of $U = K[x_1, \dots, x_n]$. Prolong these maps to maps of \mathcal{L}_n by

$$\partial_i e_{\beta, j} = e_{\alpha + \varepsilon_i, j}.$$

It is easy to see that ∂_i satisfies Leibniz rule

$$\partial_i (e_{\beta, j} e_{\gamma, s}) = (\partial_i e_{\beta, j}) e_{\gamma, s} + e_{\beta, j} (\partial_i e_{\gamma, s}),$$

for any $\beta, \gamma \in \mathbf{Z}_+^n$. So, we have constructed commuting even derivations $\partial_1, \dots, \partial_n \in \text{Der}(\mathcal{L}_n \otimes U)$ and

$$e_{\alpha, i} = \partial^\alpha e_{0, i},$$

for any $\alpha \in \mathbf{Z}_+^n, i \in I$. Here $0 = (0, \dots, 0) \in \mathbf{Z}_+^n$.

Space \mathcal{L}_n has three kinds of gradings. The first one, \mathbf{Z}^n -grading is defined by

$$||e_{\alpha,i}|| = \alpha - \varepsilon_i$$

and for other base elements are prolonged by multiplicativity,

$$||e_{\alpha,i}e_{\beta,j}\cdots e_{\gamma,s}|| = \alpha - \varepsilon_i + \beta - \varepsilon_j + \cdots + \gamma - \varepsilon_s.$$

The second grading is induced by \mathbf{Z}^n -grading. It is \mathbf{Z} -grading defined on base element $e = e_{\alpha,i}e_{\beta,j}\cdots e_{\gamma,s}$ by

$$|e| = -l(e) + |\alpha| + |\beta| + \cdots + |\gamma|.$$

The third grading is defined by length. Let $l(\xi) = s$, if ξ is a nontrivial linear combination of homogeneous base elements of length s .

Call

$$wt(e) = |\alpha| + \cdots + |\beta| - l(e)$$

weight of e . A parity on \mathcal{L}_n is defined by length. Let $\mathcal{L}_n^{[l]}$ be linear span of base elements u with $l(u) = l$. Let $\mathcal{L}_n^{[l,w]}$ be a linear span of base elements u with $l(u) = l, wt(u) = w$.

Example.

$$\mathcal{L}_n^{[1]} = \langle e_{\alpha,i} | \alpha \in \mathbf{Z}^n, i = 1, \dots, n \rangle,$$

$$\mathcal{L}_n^{[n]} = \langle e_{0,1} \cdots e_{0,n} \rangle,$$

$$\mathcal{L}_n^{[1,-1]} = \langle e_{0,i} \rangle.$$

Proposition 3.1. \mathcal{L}_n is associative, super-commutative graded algebra:

$$(uv)w = u(vw),$$

$$uv = (-1)^{q(u)q(v)}vu,$$

$$\mathcal{L}_n = \bigoplus_{l \geq 1, w \geq -n} \mathcal{L}_n^{[l,w]},$$

$$\mathcal{L}_n^{[l,w]} \mathcal{L}_n^{[l_1,w_1]} \subseteq \mathcal{L}_n^{[l+l_1,w+w_1]}.$$

for any $u, v, w \in \mathcal{L}_n$.

Note that any base element $u \in \mathcal{L}_n$ can be presented in a form $u_{-1}u_0 \cdots u_r$ where $u_s, s = -1, 0, \dots, r$ are base elements and u_s are products of generators of weight s . We say that base element $u \in \mathcal{L}_n$ has type $(l_{-1}, l_0, \dots, l_r)$, if u is a product of l_s generators of weight s , for $s = -1, 0, \dots, r$.

Lemma 3.2. *Any base element $u \in \mathcal{L}_n$ satisfy the following conditions*

$$\begin{aligned} \sum_{i \geq -1} l_i(u) &= l(u), \\ \sum_{i \geq -1} i l_i(u) &= wt(u), \\ l_i(u) &\leq n \binom{n+i}{i+1}, \quad i \geq -1. \end{aligned}$$

Proof. First two relations are reformulations of grading property of \mathcal{L}_n (proposition 3.1). As far as last two relations, they follow from the fact

$$|\{\alpha \in \mathbf{Z}_+^n \mid |\alpha| = i+1\}| = \binom{n+i}{i+1}.$$

Example. Let $u = \eta_1 \partial_1^2 \eta_2 \partial_1 \partial_2 \eta_2$. Then u is odd base element of type $(1, 0, 2)$ and $l(u) = 3, wt(u) = 1$.

Let $Diff_n$ be an algebra of differential operators on \mathcal{L}_n . It has a base consisting differential operators of a form $u \partial^\alpha$, where $\alpha \in \mathbf{Z}_+^n$ and u is a base element of \mathcal{L}_n . Endow $Diff_n$ by multiplication \cdot given by

$$u \partial^\alpha \cdot v \partial^\beta = \sum_{\gamma} \binom{\alpha}{\gamma} u \partial^\gamma v \partial^{\alpha+\beta-\gamma}.$$

Here

$$\binom{\beta}{\gamma} = \prod_{i=1}^n \binom{\beta_i + \gamma_i}{\gamma_i}.$$

Multiplication \cdot corresponds to composition of differential operators.

Endow $Diff_n$ also by two more multiplications \circ and \bullet . They are given by the following rules

$$\begin{aligned} u \partial^\alpha \circ v \partial^\beta &= \sum_{\gamma \neq 0} \binom{\alpha}{\gamma} u \partial^\gamma v \partial^{\alpha+\beta-\gamma}, \\ u \partial^\alpha \bullet v \partial^\beta &= uv \partial^{\alpha+\beta}. \end{aligned}$$

We see that

$$X \cdot Y = X \circ Y + X \bullet Y,$$

for any $X, Y \in Diff_n$.

For a base element $X = u \partial^\alpha \in Diff_n$ define *length* $l(X)$, *weight* $wt(X)$, *parity* $q(X)$ and *differential order* $\partial deg(X)$ by

$$\begin{aligned} l(X) &= l(u), \\ wt(X) &= wt(u) + |\alpha|, \\ q(X) &= l(u), \end{aligned}$$

$$\partial \deg(X) = |\alpha|.$$

Let

$$\begin{aligned} Diff_n^{[d]} &= \langle X | \partial \deg(X) = d \rangle, \\ Diff_n^{[l,w]} &= \langle X | l(X) = l, wt(X) = w \rangle, \\ Diff_n^{[l,w,d]} &= \langle X | l(X) = l, wt(X) = w, \partial \deg(X) = d \rangle. \end{aligned}$$

Denote a space of differential operators of first order $Diff_n^{[1]}$ by W_n .

For a differential operator $X = \sum_{\alpha \in \mathbf{Z}_+^n} v_\alpha \partial^\alpha \in Diff_n$, define its differential order $\deg(X)$ as maximal $|\alpha|$, such that $v_\alpha \neq 0$.

Proposition 3.3. *Space of differential operators under different multiplications have the following properties.*

The algebra $(Diff_n, \cdot)$ is associative super-algebra:

$$X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z,$$

for any $X, Y, Z \in Diff_n$. This algebra is graded,

$$\begin{aligned} Diff_n &= \oplus_{l>0, w \geq -n} Diff_n^{[l,w]}, \\ Diff_n^{[l,w]} \cdot Diff_n^{[l_1, w_1]} &\subseteq Diff_n^{[l+l_1, w+w_1]}. \end{aligned}$$

The algebra (W_n, \circ) is super-left-symmetric:

$$(X, Y, Z) = (-1)^{q(X)q(Y)}(Y, X, Z),$$

for any differential operators of first order X, Y, Z , where $(X, Y, Z) = X \circ (Y \circ Z) - (X \circ Y) \circ Z$ is associator. Moreover, super-left-symmetric rule is true for any $X, Y \in Diff_n^{[1]}, Z \in Diff_n$. This algebra is graded,

$$\begin{aligned} W_n &= \oplus_{l>0, w \geq -n} W_n^{[l,w]}, \\ W_n^{[l,w]} \circ W_n^{[l_1, w_1]} &\subseteq W_n^{[l+l_1, w+w_1]}. \end{aligned}$$

The algebra $(Diff_n, \bullet)$ is associative super-commutative:

$$\begin{aligned} X \bullet Y &= (-1)^{q(X)q(Y)} Y \bullet X, \\ X \bullet (Y \bullet Z) &= (X \bullet Y) \bullet Z, \end{aligned}$$

for any $X, Y, Z \in Diff_n$. This algebra is graded under length, weight and differential order,

$$\begin{aligned} Diff_n &= \oplus_{l>0, w \geq -n, d \geq 0} Diff_n^{[l,w,d]}, \\ Diff_n^{[l,w,d]} \bullet Diff_n^{[l_1, w_1, d_1]} &\subseteq Diff_n^{[l+l_1, w+w_1, d+d_1]}. \end{aligned}$$

Any differential operator of first order under multiplication \circ acts on $(Diff_n, \bullet)$ as a derivation:

$$X \circ (Y \bullet Z) = (X \circ Y) \bullet Z + (-1)^{q(X)q(Y)} Y \bullet (X \circ Z),$$

for any $X \in W_n, Y, Z \in Diff_n$.

Proof. Notice that natural action of W_n on \mathcal{L}_n coincides with left-symmetric product:

$$X(\eta) = X \circ \eta,$$

for any $X \in W_n, \eta \in \mathcal{L}_n$. Therefore, we have the following connection between composition and left-symmetric multiplications:

$$(X \cdot Y)(\eta) \neq (X \circ Y)(\eta)$$

but

$$(X \cdot Y)(\eta) = X \circ Y(\eta),$$

for any $X, Y \in \text{Diff}_n, \eta \in \mathcal{L}_n$. Moreover, composition of differential operators of first order can be expressed in terms of left-symmetric multiplication,

$$(X \cdot Y)(\eta) = X \circ (Y \circ \eta),$$

for any $X, Y \in W_n, \eta \in \mathcal{L}_n$. Thus,

$$(X \circ Y + X \bullet Y)(\eta) = X \circ (Y \circ \eta),$$

and

$$X \circ (Y \circ \eta) - (X \circ Y)(\eta) = (X \bullet Y)(\eta).$$

Since $X \circ Y \in W_n$, this means that

$$X \circ (Y \circ \eta) - (X \circ Y) \circ \eta = (X \bullet Y)(\eta). \quad (1)$$

for any $X, Y \in W_n, \eta \in \mathcal{L}_n$. By these facts we see that

$$\begin{aligned} ([X, Y] \cdot Z)(\eta) &= (X \cdot Y - (-1)^{q(X)q(Y)} Y \cdot X)(Z(\eta)) \\ &= (X \circ Y + X \bullet Y - (-1)^{q(X)q(Y)} Y \circ X - (-1)^{q(X)q(Y)} Y \bullet X) \circ (Z(\eta)) \\ &= (X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ (Z(\eta)). \end{aligned}$$

On the other hand

$$\begin{aligned} ([X, Y] \cdot Z)(\eta) &= (X \cdot (Y \cdot Z) - (-1)^{q(X)q(Y)} Y \cdot (X \cdot Z))(\eta) \\ &= X \circ (Y \cdot Z)(\eta) - (-1)^{q(X)q(Y)} Y \circ (X \cdot Z)(\eta) \\ &= X \circ (Y \circ Z(\eta)) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z(\eta)). \end{aligned}$$

Hence,

$$(X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ (Z(\eta)) = X \circ (Y \circ Z(\eta)) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z(\eta)).$$

In other words,

$$(X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ Z = X \circ (Y \circ Z) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z),$$

for any $X, Y \in W_n, Z \in \text{Diff}_n$.

Other statements of our proposition are evident. \square

For a base element $X = u\partial^\alpha \in \text{Diff}_n$ say that it has *type* $(l_{-1}, l_0, l_1, \dots, l_r; d)$ if u has type $(l_{-1}, l_0, \dots, l_r)$ and $|\alpha| = d$.

Example. Let $X = \eta_1 \eta_3 \partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_2 \partial_1 \partial_2 \partial_3 \eta_3 \partial_1 \partial_2$. Then X is base element of $Diff_3$ of type $(2, 3, 0, 1; 1)$, weight 2 and differential order 2.

Lemma 3.4. Any base element $X \in Diff_n$ satisfies the following conditions:

$$\begin{aligned} \sum_{i \geq -1} l_i(X) &= l(X), \\ \sum_{i \geq -1} i l_i(X) + \deg(X) &= wt(X), \\ l_i(X) &\leq n \binom{n+i}{i+1}, \quad i \geq -1. \end{aligned}$$

Proof. Follows from proposition 3.3 and Lemma 3.2. \square

Let $Diff_n^{(l_{-1}, l_0, \dots, l_r; d)}$ be a subspace of $Diff_n$ generated by base elements of type $(l_{-1}, l_0, \dots, l_r; d)$. Let

$$\begin{aligned} \tau_{(l_{-1}, l_0, \dots, l_r; d)} : Diff_n &\rightarrow Diff_n^{(l_{-1}, l_0, \dots, l_r; d)}, \\ \tau_d : Diff_n &\rightarrow Diff_n^{[d]} \end{aligned}$$

be projection maps.

Polynomial space $U = K[x_1, \dots, x_n]$ has natural gradings:

$$||x^\alpha|| = \alpha, \quad |x^\alpha| = |\alpha|.$$

It has standard base $\{x^\alpha = \prod_{i=1}^n x_i^{\alpha_i} | \alpha \in \mathbf{Z}^n\}$. These gradings on \mathcal{L}_n and U induce gradings on $\mathcal{L}_n \otimes U$.

In previous section we define parity q on $\mathcal{L}_n \otimes U$. Below we set

$$\eta_i = e_{0,i}.$$

So, instead of $e_{\alpha,i}$ we can write $\partial^\alpha \eta_i$. Then for $\eta = \partial^{\alpha_1} \eta_{i_1} \cdots \partial^{\alpha_k} \eta_{i_k}$ we have

$$l(\eta) = k.$$

We identify \mathcal{L}_n with $\mathcal{L}_n \otimes 1$ and consider \mathcal{L}_n as a subalgebra of $\mathcal{L}_n \otimes U$.

4. DIFFERENTIAL OPERATORS OF FIRST ORDER ON \mathcal{L}_n

$W_n = Diff_n^{[1]}$ has two algebraic structures. The first one, a structure of super-Lie algebra, is well-known. Let

$$[D_1, D_2] = D_1 D_2 - (-1)^{q(D_1)q(D_2)} D_2 D_1.$$

be super-commutator. Then

$$\begin{aligned} [D_1, D_2] &= -(-1)^{q(D_1)q(D_2)} [D_2, D_1], \\ [D_1, [D_2, D_3]] &= [[D_1, D_2], D_3] + (-1)^{q(D_1)q(D_2)} [D_2, [D_1, D_3]]. \end{aligned}$$

Notice that

$$q(\xi \partial_i) = q(\xi),$$

for any $\xi \in \mathcal{L}_n$. Recall that for any $D \in W_n$, corresponding adjoint operator

$$ad D : W_n \Rightarrow W_n$$

is a derivation of W_n . Therefore, W_n can be interpreted as a derivation super-Lie algebra of \mathcal{L}_n .

The second structure of algebra on W_n can be done by left-symmetric multiplication. It is less known. Define a product \circ by

$$(\xi \partial_i) \circ (\eta \partial_j) = \xi \partial_i(\eta) \partial_j.$$

Then for any $D_1, D_2, D_3 \in W_n$,

$$(D_1, D_2, D_3) = (-)^{q(D_1)q(D_2)}(D_2, D_1, D_3)$$

(left-symmetric identity). Here

$$(D_1, D_2, D_3) = D_1 \circ (D_2 \circ D_3) - (D_1 \circ D_2) \circ D_3$$

is associator.

Remark. Let $Diff_n^{(k)}$ be subspace of $Diff_n$ of order no more than k . Well known that

$$Diff_0^{(0)} = U \subset Diff_n^{(1)} \subset Diff_n^{(2)} \subset \dots$$

is an increasing filtration on $Diff_n$,

$$Diff_n^{(k)} \cdot Diff_n^{(s)} \subseteq Diff_n^{(k+s)}, \quad k, s \geq 0.$$

So, $Diff_n^{(k)}$ has a structure of algebra under composition operation, if $k = 0$, $Diff_n^{(1)}$ has an algebraic structure under commutator. One can ask about algebraic structures on $Diff_n^{(k)}$ for $k > 0$. In other words, is it possible to find some $N = N(n, k)$, such that

$$X_1, \dots, X_N \in Diff_n^{(k)} \Rightarrow s_N(X_1, \dots, X_N) \in Diff_n^{(k)}.$$

One can prove the following

Theorem 4.1. *Let $n > 1$. Then $s_{(n+1)^2} = 0$ is identity on $Diff_n^{(1)}$ and s_{n^2+2n}, s_{n^2+2n-1} are well-defined operations on $Diff_n^{(1)}$. Moreover, $s_{n^2+2n}(X_1, \dots, X_{n^2+2n}) \in Diff_n^{(0)}$, for all $X_1, \dots, X_{n^2+2n} \in Diff_n^{(1)}$.*

5. CALCULATION OF D^n

Let η_1, \dots, η_n are odd elements and

$$D = \sum_{i=1}^n \eta_i \partial_i,$$

$$F = D \circ D = \sum_{i,j=1}^n \eta_i \partial_i \eta_j \partial_j.$$

Notice that

- $D \in W_n^{[1,0]}$
- F is even element of W_n
- $l_{-1}(F) = 1, l_0(F) = 1, l_s(F) = 0, s > 0$.

Therefore,

$$D^k \in Diff_n^{[k,0]}.$$

Define left-symmetric power $D^{\circ k}$ by

$$\begin{aligned} D^{\circ k} &= D \circ D^{\circ(k-1)}, \text{ if } k > 1, \\ D^{\circ 1} &= D. \end{aligned}$$

Similarly one defines bullet power $D^{\bullet k}$ and associative power $D^{\cdot k}$. Since multiplications \cdot and \bullet are associative, in last cases $D^{\bullet k}$ and $D^{\cdot k}$ have usual properties of powers

$$\begin{aligned} D^{\bullet k} \bullet D^{\bullet s} &= D^{\bullet(k+s)}, \\ D^{\cdot k} \bullet D^{\cdot s} &= D^{\cdot(k+s)}, \end{aligned}$$

These facts are not true for left-symmetric powers. For example,

$$D \circ (D \circ D^{\circ 2}) = (D \circ D) \circ D^{\circ 2},$$

but

$$D \circ D^{\circ 2} \circ D \neq (D \circ D^{\circ 2}) \circ D.$$

Lemma 5.1. $D^{\cdot 2} = F$.

Proof.

$$D^{\cdot 2} = D \cdot D = \sum_{i,j=1}^n \eta_i \partial_i \eta_j \partial_j + \sum_{i,j=1}^n \eta_i \eta_j \partial_i \partial_j.$$

Since $\eta_i \eta_j = -\eta_j \eta_i$ and $\partial_i \partial_j = \partial_j \partial_i$, we have

$$\sum_{i,j=1}^n \eta_i \eta_j \partial_i \partial_j = 0.$$

Thus,

$$D^{\cdot 2} = \sum_{i,j=1}^n \eta_i \partial_i \eta_j \partial_j = D \circ D = F.$$

Lemma 5.2. $D^{\circ(2n)} = F^{\circ n}$ for any $n = 1, 2, 3, \dots$

Proof. We use induction on n .

If $n = 1$, then nothing is to prove.

Suppose that

$$D^{\circ(2(n-1))} = F^{\circ(n-1)}$$

for some $n > 1$. Then by definition

$$D^{\circ(2n)} = D \circ (D \circ D^{\circ(2(n-1))})$$

Since D is odd, by left-symmetric property of (W_n, \circ) (proposition 3.3)

$$(D, D, G) =$$

for any $G \in W_n$. Thus,

$$D \circ (D \circ G) = (D \circ D) \circ G.$$

Therefore,

$$D^{\circ(2n)} = (D \circ D) \circ D^{\circ(2(n-1))}.$$

By inductive suggestion,

$$D^{\circ(2n)} = F \circ F^{\circ(n-1)} = F^{\circ n}.$$

Lemma 5.3. $F \circ F^{\bullet k} = k F^{\bullet(k-1)} \bullet F^{\circ 2}$

Proof. Since $F \in W_n$ is even derivation, any any left-symmetric multiplication operator acts on $(Diff_n, \bullet)$ as a super-derivation (proposition 3.3) we have

$$F \circ (F^{\bullet} F) = (F \circ F) \bullet F + F \bullet (F \circ F).$$

By commutativity of bullet-multiplication this means that

$$F \circ F^{\bullet 2} = 2F \bullet F^{\circ 2}.$$

Easy induction on k based on a such arguments shows that our lemma is true in general case.

Lemma 5.4. $D^4 = F^{\circ 2} + F^{\bullet 2}$

Proof. By Lemma 5.1 and by associativity of \circ ,

$$D^4 = D^2 \cdot D^2 = F \cdot F = F \circ F + F \bullet F.$$

Lemma 5.5. $D^6 = F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}$.

Proof. By Lemma 5.1 and Lemma 5.4,

$$\begin{aligned} D^6 &= D^2 \cdot D^4 = D^2 \circ D^4 + D^2 \bullet D^4 \\ &= F \circ (F^{\circ 2} + F^{\bullet 2}) + F \bullet (F^{\circ 2} + F^{\bullet 2}) \\ &= F^{\circ 3} + F \circ F^{\bullet 2} + F \bullet F^{\circ 2} + F^{\bullet 3}. \end{aligned}$$

Thus by Lemma 5.3,

$$D^6 = F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}.$$

Lemma 5.6. $D^8 = F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4}.$

Proof. By Lemma 5.3

$$F \circ F^{\bullet 3} = 3F^{\circ 2} \bullet F^{\bullet 2}.$$

Therefore, by Lemma 5.1, Lemma 5.5

$$\begin{aligned} D^8 &= D^2 \cdot D^6 = D^2 \circ D^6 + D^2 \bullet D^6 \\ &= F \circ (F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}) \\ &\quad + F \bullet (F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}) \\ &= F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 3F \bullet F^{\circ 3} + 3F^{\circ 2} \bullet F^{\bullet 2} \\ &\quad + F \bullet F^{\circ 3} + 3F^{\bullet 2} \bullet F^{\circ 2} + F^{\bullet 4} \\ &= F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4}. \end{aligned}$$

Lemma 5.7.

$$\begin{aligned} D^{10} &= F^{\circ 5} \\ &\quad + 5(F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3})) \\ &\quad + 5(2F^{\circ 3} \bullet F^{\bullet 2} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2}) \\ &\quad + 4F^{\circ 2} \bullet F^{\bullet 3} + 6F^{\circ 2} \bullet F^{\bullet 3} + F^{\bullet 5} \end{aligned}$$

Proof. By Lemma 5.6 and Lemma 5.3

$$\begin{aligned} D^{10} &= D^2 \circ D^8 = D^2 \circ D^8 + D^2 \bullet D^8 \\ &= F \circ (F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4}) \\ &\quad + F \bullet (F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4}) \\ &= F^{\circ 5} + 3F \circ (F^{\circ 2} \bullet F^{\circ 2}) + 4F \circ (F \bullet F^{\circ 3}) + 6F \circ (F^{\circ 2} \bullet F^{\bullet 2}) + F \circ (F^{\bullet 4}) \\ &\quad + F \bullet F^{\circ 4} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2} + 4F^{\bullet 2} \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 3} + F^{\bullet 5}) \\ &= F^{\circ 5} + 6F^{\circ 2} \bullet F^{\circ 3} + 4F^{\circ 2} \bullet F^{\circ 3} + 4F \bullet F^{\circ 4} \\ &\quad + 6F^{\circ 3} \bullet F^{\bullet 2} + 12F^{\circ 2} \bullet F^{\circ 2} \bullet F + 4F^{\circ 2} \bullet F^{\bullet 3} \\ &\quad + F \bullet F^{\circ 4} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2} + 4F^{\bullet 2} \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 3} + F^{\bullet 5}) \\ &= F^{\circ 5} \end{aligned}$$

$$\begin{aligned}
& +10F^{\circ 2} \bullet F^{\circ 3} + 5F \bullet F^{\circ 4} \\
& +10F^{\circ 3} \bullet F^{\bullet 2} + 15F \bullet F^{\circ 2} \bullet F^{\circ 2} \\
& +4F^{\circ 2} \bullet F^{\bullet 3} + 6F^{\circ 2} \bullet F^{\bullet 3} \\
& +F^{\bullet 5}.
\end{aligned}$$

By Lemma 5.3,

$$F \circ (F^{\circ 3} \bullet F) = F^{\circ 4} \bullet F + F^{\circ 3} \bullet F^{\circ 2}.$$

Thus

$$\begin{aligned}
& 2F^{\circ 2} \bullet F^{\circ 3} + F \bullet F^{\circ 4} = \\
& F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3}).
\end{aligned}$$

Our lemma is proved.

Lemma 5.8. *For any $G \in Diff_n$,*

$$F \circ \left(\prod_{r=1}^n \eta_r G \right) = 0.$$

Proof. We have

$$\begin{aligned}
& \eta_i \partial_i (\eta_j) \partial_j (\eta_1 \cdots \eta_n) \\
& = \sum_{s=1}^n \xi_s,
\end{aligned}$$

where

$$\xi_s = \eta_i \partial_i \eta_j \eta_1 \cdots \eta_{s-1} \partial_j (\eta_s) \eta_{s+1} \cdots \eta_n.$$

If $s \neq i$, then

$$\xi_s = \pm \eta_i \eta_i \xi_{i,s},$$

where

$$\xi_{i,s} = \partial_i \eta_j \partial_j \eta_s \prod_{r \neq i,s} \eta_r.$$

Since $\eta_i \eta_i = 0$, this means that

$$\xi_s = 0,$$

if $s \neq i$. If $s = i$, then

$$\xi_s = \pm \eta_i \partial_i \eta_j \partial_j \eta_i \prod_{r \neq i} \eta_r = \partial_i \eta_j \partial_j \eta_i \left(\prod_r \eta_r \right).$$

We have

$$\sum_{i,j=1}^n \partial_i \eta_j \partial_j \eta_i = \theta_1 + \theta_2 + \theta_3,$$

where

$$\begin{aligned}
\theta_1 &= \sum_{i < j} \partial_i \eta_j \partial_j \eta_i, \\
\theta_2 &= \sum_i \partial_i \eta_i \partial_i \eta_i,
\end{aligned}$$

$$\theta_3 = \sum_{i>j} \partial_i \eta_j \partial_j \eta_i,$$

Since elements $\partial_i \eta_j$ and $\partial_j \eta_i$ are odd,

$$\theta_1 + \theta_3 = 0, \quad \theta_2 = 0.$$

Thus,

$$F \circ \left(\prod_{r=1}^n \eta_r G \right) = \left(\sum_{i,j=1}^n \partial_i \eta_j \partial_j \eta_i \right) \prod_r \eta_r G = 0.$$

Let

$$Diff_n^{[s]} = \langle u \partial^\alpha | u \in \mathcal{L}_n, \alpha \in \Gamma_n, |\alpha| = s \rangle$$

be a space of differential operators of order s and

$$\tau_s : Diff_n \rightarrow Diff_n^{[s]}$$

be projection map.

Lemma 5.9. *If $n = 3$, $D = \sum_{i=1}^n u_i \partial_i$, and u_i are odd, then*

$$\tau_1 D = F^{\circ 5},$$

$$\tau_2 D^{10} = 5(F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3})),$$

$$\tau_3 D^{10} = 5(2F^{\circ 3} \bullet F^{\bullet 2} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2}),$$

$$\tau_s D^{10} = 0, \quad s > 3.$$

Proof. Follows from Lemma 5.7 and from the fact that $F^{\bullet s} = 0$, if $s > n$.

Conclusion. To find D^{10} we need to calculate $F^{\circ s}$, for $s = 1, 2, 3$ and $F^{\bullet 2}$.

6. SECOND BULLET-POWER OF F

The following calculations are not difficult.

$$F_{\eta_1 \eta_2; \partial_1^2}^{\bullet 2} = -2\eta_1 \eta_2 \partial_1 \eta_1 \partial_2 \eta_1,$$

$$F_{\eta_1 \eta_3; \partial_1^2}^{\bullet 2} = -\eta_1 \eta_3 2\partial_1 \eta_1 \partial_3 \eta_1 \partial_1^2,$$

$$F_{\eta_2 \eta_3; \partial_1^2}^{\bullet 2} = -2\eta_2 \eta_3 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1^2,$$

7. SECOND LEFT-SYMMETRIC POWER OF F

It is not hard to obtain the following results.

$$F_{\eta_1; \partial_1}^{\circ 2} =$$

$$\eta_1(-2\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2 - 2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3 + \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2$$

$$- \partial_2\eta_1\partial_3\eta_2\partial_1\eta_3 + \partial_3\eta_1\partial_1\eta_2\partial_2\eta_3 + \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3)\partial_1,$$

$$F_{\eta_2; \partial_1}^{\circ 2} =$$

$$\eta_2(-\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2 - \partial_1\eta_1\partial_3\eta_1\partial_2\eta_3 - \partial_2\eta_1\partial_3\eta_1\partial_1\eta_3$$

$$- \partial_2\eta_1\partial_3\eta_2\partial_2\eta_3 + \partial_3\eta_1\partial_2\eta_2\partial_2\eta_3 + \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3)\partial_1,$$

$$F_{\eta_3; \partial_1}^{\circ 2} =$$

$$\eta_3(-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2 - \partial_1\eta_1\partial_3\eta_1\partial_3\eta_3 - \partial_2\eta_1\partial_2\eta_2\partial_3\eta_2$$

$$+ \partial_2\eta_1\partial_3\eta_1\partial_1\eta_2 - \partial_2\eta_1\partial_3\eta_2\partial_3\eta_3 + \partial_3\eta_1\partial_3\eta_2\partial_2\eta_3)\partial_1,$$

$$F_{\eta_1\eta_2; \partial_1}^{\circ 2} =$$

$$\eta_1\eta_2(-\partial_1\eta_1\partial_1\partial_2\eta_1 - \partial_1\eta_2\partial_2^2\eta_1 - \partial_1\eta_3\partial_2\partial_3\eta_1$$

$$+ \partial_2\eta_1\partial_1^2\eta_1 + \partial_2\eta_2\partial_1\partial_2\eta_1 + \partial_2\eta_3\partial_1\partial_3\eta_1)\partial_1,$$

$$F^2(\eta_1\eta_3; \partial_1) =$$

$$\eta_1\eta_3(-\partial_1\eta_1\partial_1\partial_3\eta_1 - \partial_1\eta_2\partial_2\partial_3\eta_1 - \partial_1\eta_3\partial_3^2\eta_1$$

$$+ \partial_3\eta_1\partial_1^2\eta_1 + \partial_3\eta_2\partial_1\partial_2\eta_1 + \partial_3\eta_3\partial_1\partial_3\eta_1)\partial_1,$$

$$F_{\eta_2\eta_3; \partial_1}^{\circ 2} =$$

$$\eta_2\eta_3(-\partial_2\eta_1\partial_1\partial_3\eta_1 - \partial_2\eta_2\partial_2\partial_3\eta_1 - \partial_2\eta_3\partial_3^2\eta_1$$

$$+ \partial_3\eta_1\partial_1\partial_2\eta_1 + \partial_3\eta_2\partial_2^2\eta_1 + \partial_3\eta_3\partial_2\partial_3\eta_1)\partial_1.$$

8. THIRD LEFT-SYMMETRIC POWER OF F

In this section we give results of some calculations concerning $F^{\circ 3} = F \circ (F \circ F)$

$$F_{\eta_1; \partial_1}^{\circ 3} =$$

$$\begin{aligned} & \eta_1 (2\partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 + 2\partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_1 \eta_3 - 6\partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_1 \eta_3 \\ & - 2\partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_2 \partial_1 \eta_3 \partial_3 \eta_3 - 2\partial_1 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_2 \eta_3 - 2\partial_1 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3 \\ & + 2\partial_1 \eta_1 \partial_3 \eta_1 \partial_3 \eta_2 \partial_1 \eta_3 \partial_2 \eta_3 - 2\partial_2 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 + \partial_2 \eta_1 \partial_1 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3 \\ & - \partial_2 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_1 \eta_3 \partial_3 \eta_3 + 2\partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_1 \eta_3 \partial_3 \eta_3 - 2\partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_1 \eta_3 \\ & + \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3 - \partial_3 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_1 \eta_3 \partial_2 \eta_3 - 2\partial_3 \eta_1 \partial_3 \eta_2 \partial_1 \eta_3 \partial_2 \eta_3 \partial_3 \eta_3) \partial_1 \\ & \text{(all together 15 terms)} \end{aligned}$$

$$F_{\eta_2; \partial_1}^{\circ 3} =$$

$$\begin{aligned} & \eta_2 (2\partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 - 2\partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_3 - 2\partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_2 \eta_2 \partial_1 \eta_3 \\ & - \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3 - \partial_1 \eta_1 \partial_3 \eta_1 \partial_2 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3 - 2\partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_2 \eta_3 \\ & + \partial_2 \eta_1 \partial_3 \eta_1 \partial_2 \eta_2 \partial_1 \eta_3 \partial_3 \eta_3 + \partial_2 \eta_1 \partial_3 \eta_1 \partial_3 \eta_2 \partial_1 \eta_3 \partial_2 \eta_3) \partial_1. \\ & \text{(all together 8 terms)} \end{aligned}$$

$$F_{\eta_3; \partial_1}^{\circ 3} =$$

$$\begin{aligned} & \eta_3 (\partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_3 \eta_3 - 2\partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_3 \eta_3 - 2\partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_3 \eta_2 \partial_1 \eta_3 \\ & + \partial_1 \eta_1 \partial_3 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 - 2\partial_1 \eta_1 \partial_3 \eta_1 \partial_3 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3 - \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_3 \eta_3 \\ & - \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 + 2\partial_2 \eta_1 \partial_3 \eta_1 \partial_3 \eta_2 \partial_1 \eta_3 \partial_3 \eta_3) \partial_1. \\ & \text{(all together 8 terms)} \end{aligned}$$

$$F_{\eta_1 \eta_2; \partial_1}^{\circ 3} =$$

$$\begin{aligned} & \eta_1 \eta_2 (3\partial_1 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_2^2 \eta_1 + 3\partial_1 \eta_1 \partial_1 \eta_2 \partial_2 \eta_3 \partial_2 \partial_3 \eta_1 + 4\partial_1 \eta_1 \partial_1 \eta_3 \partial_2 \eta_3 \partial_3^2 \eta_1 \\ & - \partial_1 \eta_1 \partial_1 \eta_3 \partial_3 \eta_3 \partial_2 \partial_3 \eta_1 - 2\partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta_2 \partial_1 \partial_2 \eta_1 + \partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta_2 \partial_2^2 \eta_2 \\ & - 5\partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta_3 \partial_1 \partial_3 \eta_1 + \partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta_3 \partial_2 \partial_3 \eta_2 - 2\partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_2 \partial_1 \partial_2 \eta_2 \\ & + \partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_2 \partial_1^2 \eta_1 - 3\partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_3 \partial_1 \partial_3 \eta_2 + 3\partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1^2 \eta_3 \\ & + \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_2 \partial_1 \partial_2 \eta_3 - 4\partial_1 \eta_1 \partial_2 \eta_2 \partial_1 \eta_3 \partial_2 \partial_3 \eta_1 - 2\partial_1 \eta_1 \partial_2 \eta_2 \partial_2 \eta_3 \partial_1 \partial_3 \eta_1 \\ & - 2\partial_1 \eta_1 \partial_2 \eta_3 \partial_3 \eta_3 \partial_1 \partial_3 \eta_1 + \partial_1 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2^2 \eta_3 + 3\partial_1 \eta_1 \partial_3 \eta_1 \partial_1 \eta_3 \partial_1 \partial_2 \eta_1 \\ & + \partial_1 \eta_1 \partial_3 \eta_1 \partial_1 \eta_3 \partial_2 \partial_3 \eta_3 - 3\partial_1 \eta_1 \partial_3 \eta_1 \partial_2 \eta_2 \partial_1 \partial_2 \eta_3 + \partial_1 \eta_1 \partial_3 \eta_1 \partial_2 \eta_3 \partial_1 \partial_2 \eta_2 \\ & - 3\partial_1 \eta_1 \partial_3 \eta_1 \partial_2 \eta_3 \partial_1 \partial_3 \eta_3 + \partial_1 \eta_1 \partial_3 \eta_1 \partial_2 \eta_3 \partial_1^2 \eta_1 + \partial_1 \eta_1 \partial_3 \eta_1 \partial_3 \eta_3 \partial_1 \partial_2 \eta_3) \end{aligned}$$

$$\begin{aligned}
& +\partial_1\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_1 + 2\partial_1\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 - 3\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1 \\
& - 3\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 + 3\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_1 - 4\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 \\
& + 2\partial_2\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 - 2\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\partial_2\eta_1 + \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2^2\eta_2 \\
& + 2\partial_2\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2 - \partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_3 + 2\partial_2\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_2 \\
& + 3\partial_2\eta_1\partial_1\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 + \partial_2\eta_1\partial_2\eta_2\partial_1\eta_3\partial_1\partial_3\eta_1 - \partial_2\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_2 \\
& - \partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1\partial_3\eta_2 + \partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_2\eta_3 + 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\partial_2\eta_3 \\
& + 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_3\eta_3 - 4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1^2\eta_1 - \partial_2\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1^2\eta_2 \\
& - \partial_2\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1^2\eta_3 - 2\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_1\partial_2\eta_1 + \partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_3 \\
& - \partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_3 - \partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1^2\eta_1 + 2\partial_2\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1 \\
& + 2\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 + \partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 - \partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_1 \\
& - 2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 - 2\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2^2\eta_1 + 2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2^2\eta_3 \\
& - 2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 + 2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_3 - \partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2^2\eta_2 \\
& - \partial_3\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2^2\eta_3 - 4\partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_1\partial_3\eta_1 - \partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_2 \\
& + 2\partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_3 - \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1 - \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_3 \\
& + 3\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_1\partial_2\eta_1 - 2\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_3 + \partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1\partial_2\eta_2 \\
& + \partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1^2\eta_1 + \partial_3\eta_1\partial_2\eta_2\partial_3\eta_3\partial_1\partial_2\eta_3 + \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_3 \\
& + \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1^2\eta_1 - 3\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_1 - \partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_1 \\
& + \partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1)\partial_1.
\end{aligned}$$

(all together 76 terms)

$$\begin{aligned}
& F_{\eta_1\eta_3;\partial_1}^{\circ 3} = \\
& \eta_1\eta_3(-\partial_1\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\partial_3\eta_1 - \partial_1\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3^2\eta_1 + 4\partial_1\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_1 \\
& + 4\partial_1\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2\partial_3\eta_1 + 3\partial_1\eta_1\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_1 + 3\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_1\partial_3\eta_1 \\
& + \partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\partial_3\eta_2 + \partial_1\eta_1\partial_2\eta_1\partial_1\eta_3\partial_3^2\eta_2 + \partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_1\partial_3\eta_2 \\
& - 3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1^2\eta_2 - 3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\partial_2\eta_2 + \partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\partial_3\eta_3 \\
& + \partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1^2\eta_1 - 3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_3\partial_1\partial_3\eta_2 + 2\partial_1\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_2\eta_1 \\
& - 5\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\partial_2\eta_1 + \partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\partial_3\eta_3 - 2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_3\eta_1 \\
& + \partial_1\eta_1\partial_3\eta_1\partial_1\eta_3\partial_3^2\eta_3 + \partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1\partial_3\eta_2 - 3\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\partial_2\eta_3 \\
& - 2\partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1\partial_3\eta_3 + \partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1^2\eta_1 - 3\partial_1\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 \\
& - 2\partial_1\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1 + 2\partial_1\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1 - \partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3^2\eta_1 \\
& + 4\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_1 + 2\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_1 + \partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 \\
& - 3\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1 + 2\partial_1\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_1 + 2\partial_2\eta_1\partial_1\eta_2\partial_1\eta_3\partial_3^2\eta_1 \\
& - \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\partial_3\eta_1 - \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\partial_3\eta_2 - 4\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_1\partial_2\eta_1 \\
& - \partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\partial_3\eta_3 + 2\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_2 - 3\partial_2\eta_1\partial_1\eta_2\partial_3\eta_3\partial_1\partial_3\eta_1
\end{aligned}$$

$$\begin{aligned}
& +2\partial_2\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2\partial_3\eta_2 + 2\partial_2\eta_1\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_2 + \partial_2\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3^2\eta_2 \\
& -\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_2\eta_2 - \partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1^2\eta_1 - \partial_2\eta_1\partial_2\eta_2\partial_3\eta_3\partial_1\partial_3\eta_2 \\
& -2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\partial_2\eta_2 + 4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1^2\eta_1 - 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_3\eta_2 \\
& +\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1^2\eta_2 + \partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1^2\eta_3 + 2\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_1\partial_3\eta_1 \\
& -2\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_2 + \partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3^2\eta_3 - \partial_2\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1\partial_3\eta_3 \\
& -\partial_2\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1^2\eta_1 + 3\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 - \partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1 \\
& -\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1 - 2\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 + 3\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\partial_2\eta_1 \\
& +2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1 - \partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2 + 2\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_3 \\
& -\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 + \partial_3\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2\partial_3\eta_3 - \partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_2 \\
& -2\partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 + \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_3 - \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_2 \\
& -2\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_3 + \partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_2 + \partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1^2\eta_1 \\
& +\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_3 + 3\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1 - 3\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 \\
& +2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1)\partial_1.
\end{aligned}$$

(all together 76 terms)

$$\begin{aligned}
& F_{\eta_2\eta_3;\partial_1}^{\circ 3} = \\
& \eta_2\eta_3(\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\partial_3\eta_1 + \partial_1\eta_1\partial_2\eta_1\partial_1\eta_3\partial_3^2\eta_1 + \partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_1\partial_3\eta_1 \\
& +\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_2\partial_3\eta_2 + \partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_3^2\eta_2 - \partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\partial_2\eta_2 \\
& -\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\partial_3\eta_3 - 2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1^2\eta_1 - \partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\partial_2\eta_1 \\
& -\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_2^2\eta_2 - \partial_1\eta_1\partial_2\eta_1\partial_3\eta_3\partial_2\partial_3\eta_2 + 2\partial_1\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_1 \\
& +2\partial_1\eta_1\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_1 + 2\partial_1\eta_1\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 - \partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2^2\eta_1 \\
& -\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3\partial_2\partial_3\eta_1 + \partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_2\partial_3\eta_3 + \partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1\partial_3\eta_1 \\
& +\partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_3^2\eta_3 - \partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2^2\eta_3 - \partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1\partial_2\eta_1 \\
& -\partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_2\partial_3\eta_3 - 2\partial_1\eta_1\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1 - \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\partial_3\eta_1 \\
& +2\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_1 + \partial_2\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2\partial_3\eta_1 + \partial_2\eta_1\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_1 \\
& +\partial_2\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3^2\eta_1 + \partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3^2\eta_2 - 3\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_2\eta_1 \\
& -\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\partial_3\eta_3 + \partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_2 - 2\partial_2\eta_1\partial_2\eta_2\partial_3\eta_3\partial_1\partial_3\eta_1 \\
& +\partial_2\eta_1\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_2 + 2\partial_2\eta_1\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_2 - 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\partial_2\eta_1 \\
& -2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_3\eta_1 - \partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\partial_2\eta_2 + 2\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1^2\eta_1 \\
& -\partial_2\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1\partial_3\eta_2 - \partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\partial_2\eta_3 - \partial_2\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1\partial_3\eta_3 \\
& +2\partial_2\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1^2\eta_1 - 2\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 + \partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1 \\
& -2\partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2 + \partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3^2\eta_3 + \partial_2\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1 \\
& -\partial_2\eta_1\partial_3\eta_2\partial_3\eta_3\partial_2\partial_3\eta_3 + \partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 - \partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1 \\
& +\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_1 - \partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2^2\eta_1 - 2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1
\end{aligned}$$

$$\begin{aligned}
& +\partial_3\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2^2\eta_1 - 2\partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1 + \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 \\
& +\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 + \partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1 - \partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2 \\
& +2\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_3 - 2\partial_3\eta_1\partial_2\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1 + \partial_3\eta_1\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_3 \\
& -3\partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 - \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2 + \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_3 \\
& +\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 - 2\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_3 + \partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_2 \\
& +\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_3 - \partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1)\partial_1.
\end{aligned}$$

(all together 71 terms)

9. QUADRATIC DIFFERENTIAL PART OF D^{10}

For $G \in Diff_n$ denote by $G_{\eta_{i_1}\dots\eta_{i_k};\partial^\alpha}$ projection to subspace of $Diff_n$ generated by differential operators of the form $\eta_{i_1}\dots\eta_{i_k}\prod_{s,\beta\neq 0}\partial^\beta u_s\partial^\alpha$. For example, if

$$\begin{aligned}
G = & -5\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_3^2\eta_3\partial_1 + \eta_1\eta_3\partial_2\eta_1\partial_1\eta_3\partial_1\partial_2^3\eta_3\partial_3^2 \\
& -\eta_1\eta_3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_1\partial_2\partial_3\eta_3\partial_2 + 9\eta_1\eta_3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_1\partial_2\partial_3\eta_3\partial_2 \\
& -7\eta_1\eta_3\partial_1\eta_1\partial_3\eta_2\partial_1\eta_3\partial_1^2\partial_2\partial_3^2\eta_3\partial_3^2,
\end{aligned}$$

then

$$G_{\eta_1\eta_3;\partial_3^2} = \eta_1\eta_3\partial_2\eta_1\partial_1\eta_3\partial_1\partial_2^3\eta_3\partial_3^2 - 7\eta_1\eta_3\partial_1\eta_1\partial_3\eta_2\partial_1\eta_3\partial_1^2\partial_2\partial_3^2\eta_3\partial_3^2.$$

Lemma 9.1. $F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3}) = 0$.

Proof. Let

$$Q = F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3}).$$

It is enough to prove that ∂_1^2 -part of Q is equal to 0. Then by symmetry ∂_2^2 -, ∂_3^2 -parts of Q should be 0, and $\partial_1\partial_2$ -, $\partial_1\partial_3$ -, $\partial_2\partial_3$ -parts of G also will vanish.

Let us show how to calculate $\eta_1\eta_2\eta_3\partial_1^2$ -part of Q .

Notice that $\eta_1\eta_2\eta_3\partial_1^2$ -part of $F^{\circ 2} \bullet F^{\circ 3}$, denote it by G_1 , is equal to

$$\begin{aligned}
G_1 = & F_{\eta_1;\partial_1}^{\circ 2} \bullet F_{\eta_2\eta_3;\partial_1}^{\circ 3} + F_{\eta_2;\partial_1}^{\circ 2} \bullet F_{\eta_1\eta_3;\partial_1}^{\circ 3} + F_{\eta_3;\partial_1}^{\circ 2} \bullet F_{\eta_1\eta_2;\partial_1}^{\circ 3} + \\
& F_{\eta_1\eta_2;\partial_1}^{\circ 2} \bullet F_{\eta_3;\partial_1}^{\circ 3} + F_{\eta_1\eta_3;\partial_1}^{\circ 2} \bullet F_{\eta_2;\partial_1}^{\circ 3} + F_{\eta_2\eta_3;\partial_1}^{\circ 2} \bullet F_{\eta_1;\partial_1}^{\circ 3}.
\end{aligned}$$

Using results of sections 7, 8 we obtain that

$$\begin{aligned}
& F_{\eta_1;\partial_1}^{\circ 2} \bullet F_{\eta_2\eta_3;\partial_1}^{\circ 3} + F_{\eta_2;\partial_1}^{\circ 2} \bullet F_{\eta_1\eta_3;\partial_1}^{\circ 3} + F_{\eta_3;\partial_1}^{\circ 2} \bullet F_{\eta_1\eta_2;\partial_1}^{\circ 3} \\
= & \eta_1\eta_2\eta_3(-5\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 + 14\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1 \\
& -\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_1 - 8\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 \\
& +9\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1 + 4\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1
\end{aligned}$$

[illegible]

Similarly,

[illegible]

$$G_1 =$$

$$\begin{aligned} & \eta_1 \eta_2 \eta_3 (-5 \partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3 \partial_3^2 \eta_1 + 10 \partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 \partial_2 \partial_3 \eta_1 \\ & - 5 \partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3 \partial_2 \partial_3 \eta_1 + 5 \partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_1 \eta_3 \partial_2 \eta_3 \partial_3^2 \eta_1 \\ & + 5 \partial_1 \eta_1 \partial_2 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_1 \eta_3 \partial_3 \eta_3 \partial_2 \partial_3 \eta_1 + 4 \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_2 \eta_3 \partial_1 \partial_3 \eta_1 \\ & - 6 \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_2 \eta_3 \partial_2 \partial_3 \eta_2 - \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_2 \eta_3 \partial_3^2 \eta_3 \\ & + 10 \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_3 \eta_2 \partial_2^2 \eta_3 - \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_3 \eta_3 \partial_1 \partial_2 \eta_1 \\ & + 4 \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_3 \eta_3 \partial_2 \partial_3 \eta_3 - \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_1 \eta_2 \partial_2 \eta_2 \partial_3 \eta_3 \partial_2^2 \eta_2 \end{aligned}$$

[illegible]

Now calculate $\eta_1\eta_2\eta_3\partial_1^2$ -part of $F \circ (F \bullet F^{\circ 3})$. Set $G = F \bullet F^{\circ 3}$. It is easy to see that

$$G_{\eta_1\eta_2;\partial_1^2} = F_{\eta_1;\partial_1} \bullet F_{\eta_2;\partial_1}^{\circ 3} + F_{\eta_2;\partial_1} \bullet F_{\eta_1;\partial_1}^{\circ 3},$$

$$G_{\eta_1\eta_3;\partial_1^2} = F_{\eta_1;\partial_1} \bullet F_{\eta_3;\partial_1}^{\circ 3} + F_{\eta_3;\partial_1} \bullet F_{\eta_1;\partial_1}^{\circ 3},$$

$$G_{\eta_2\eta_3;\partial_1^2} = F_{\eta_2;\partial_1} \bullet F_{\eta_3;\partial_1}^{\circ 3} + F_{\eta_3;\partial_1} \bullet F_{\eta_2;\partial_1}^{\circ 3}.$$

By results of section 8,

$$G_{\eta_1\eta_2;\partial_1^2} = \eta_1\eta_2 H_{\eta_1\eta_2} \partial_1^2,$$

$$G_{\eta_1\eta_3;\partial_1^2} = \eta_1\eta_3 H_{\eta_1\eta_3} \partial_1^2,$$

$$G_{\eta_2\eta_3;\partial_1^2} = \eta_2\eta_3 H_{\eta_2\eta_3} \partial_1^2,$$

where,

$$\begin{aligned} H_{\eta_1\eta_2} = & 4\partial_1\eta_1\partial_1\eta_2\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1 - 2\partial_1\eta_1\partial_1\eta_2\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_3 \\ & - \partial_1\eta_1\partial_1\eta_3\partial_2\eta_1\partial_2\eta_2\partial_3\eta_1\partial_3\eta_3 + 3\partial_1\eta_1\partial_1\eta_3\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2 \\ & + \partial_1\eta_2\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1\partial_3\eta_3 + \partial_1\eta_3\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2 \\ & + 2\partial_1\eta_3\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3, \end{aligned}$$

$$\begin{aligned} H_{\eta_1\eta_3} = & -\partial_1\eta_1\partial_1\eta_2\partial_2\eta_1\partial_2\eta_2\partial_3\eta_1\partial_3\eta_3 + 3\partial_1\eta_1\partial_1\eta_2\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2 \\ & - 2\partial_1\eta_1\partial_1\eta_3\partial_2\eta_1\partial_2\eta_2\partial_3\eta_1\partial_3\eta_2 + 4\partial_1\eta_1\partial_1\eta_3\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3 \\ & - 2\partial_1\eta_2\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2 - \partial_1\eta_2\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3 \\ & - \partial_1\eta_3\partial_2\eta_1\partial_2\eta_2\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3, \end{aligned}$$

$$\begin{aligned} H_{\eta_2\eta_3} = & -3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2 - 3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3. \end{aligned}$$

Thus, $\eta_1\eta_2\eta_3\partial_1^2$ -part of $F \circ G$ by Lemma 5.8 is equal to $\eta_1\eta_2\eta_3\partial_1^2$ -part of

$$F \circ (G_{\eta_1\eta_2;\partial_1^2} + G_{\eta_1\eta_3;\partial_1^2} + G_{\eta_2\eta_3;\partial_1^2}).$$

So, $\eta_1\eta_2\eta_3\partial_1^2$ -part of $F \circ G$ is equal to

$$\eta_1\eta_2\eta_3(\partial_1(D) \circ H_{\eta_2\eta_3;\partial_1^2} + \partial_2(D) \circ H_{\eta_1\eta_3;\partial_1^2} + \partial_3(D) \circ H_{\eta_1\eta_2;\partial_1^2})\partial_1^2$$

Calculations show that this expression is equal to $-G_1$. So, we obtain that $\eta_1\eta_2\eta_3\partial_1^2$ -part of $F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3})$ is equal 0.

Similar calculations show that sums of $\eta_i\eta_j\partial_1^2$ -parts of $F^{\circ 2} \bullet F^{\circ 3}$ and $F \circ (F \bullet F^{\circ 3})$ are also vanish, if $i \neq j$. So, we have established that

$$F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3}) = 0.$$

10. QUBIC DIFFERENTIAL PART OF D^{10}

In this section we use denotions and results of calculations of section 9.

Lemma 10.1. $F^{\circ 3} \bullet F^{\bullet 2} = 0$

Proof. Recall that $G = F \bullet F^{\circ 3}$. By associativity and super-commutativity of bullet-multiplication (proposition ??) we have

$$F^{\circ 3} \bullet (F \bullet F) = F \bullet (F \bullet F^{\circ 3}).$$

So, $\eta_1\eta_2\eta_3\partial_1^2$ -part of $F^{\circ 3} \bullet F^{\bullet 2}$ is equal to

$$\begin{aligned} & F \bullet (\eta_1\eta_2H_{\eta_1\eta_2} + \eta_1\eta_3H_{\eta_1\eta_3} + \eta_2\eta_3H_{\eta_2\eta_3})\partial_1^2 \\ &= \eta_1\eta_2\eta_3(\partial_1\eta_1\partial_1 \bullet H_{\eta_2\eta_3}\partial_1^2 - \partial_2\eta_1\partial_1 \bullet H_{\eta_1\eta_3}\partial_1^2 + \partial_3\eta_1\partial_1 \bullet H_{\eta_1\eta_2}\partial_1^2). \end{aligned}$$

By results of section 9 it is easy to obtain that

$$\partial_1\eta_1H_{\eta_2\eta_3} = 0,$$

$$\partial_2\eta_1H_{\eta_1\eta_3} = 0,$$

$$\partial_3\eta_1H_{\eta_1\eta_2} = 0.$$

So, $\eta_1\eta_2\eta_3\partial_1^3$ -part of $F^{\circ 3} \bullet F^{\bullet 2}$ is 0. Since number of bullets is two, $\eta_i\eta_j\partial_1^3$ -parts and $\eta_i\partial_1^3$ -parts of $F^{\circ 3} \bullet F^{\bullet 2}$ are also 0. So, ∂_1^3 -part of $F^{\circ 3} \bullet F^{\bullet 2}$ vanishes. By symmetry ∂^α -part of $F^{\circ 3} \bullet F^{\bullet 2}$ vanishes also for any $\alpha \in \Gamma_3$, such that $|\alpha| = 3$. Lemma is proved.

Lemma 10.2. $F \bullet F^{\circ 2} \bullet F^{\circ 2} = 0$.

Proof. Let $R = F^{\circ 2} \bullet F^{\circ 2}$. We use calculations on $F^{\circ 2}$ (section 7) to obtain

$$\begin{aligned} & R_{\eta_1\eta_2;\partial_1^2} \\ &= \eta_1\eta_2(8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3 - 4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3 \\ &\quad + 2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3 + 6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3 \\ &\quad - 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3 + 2\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3 + \\ &\quad 4\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3)\partial_1^2, \end{aligned}$$

$$\begin{aligned} & R_{\eta_1\eta_3;\partial_1^2} \\ &= \eta_1\eta_3(-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3 - 6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3 \\ &\quad - 4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3 + 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3 \\ &\quad - 4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3 + 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3 \\ &\quad - 2\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3)\partial_1^2, \end{aligned}$$

$$R_{\eta_2\eta_3;\partial_1^2}$$

$$= \eta_2 \eta_3 (-6 \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_2 \eta_2 \partial_3 \eta_2 \partial_2 \eta_3 + 6 \partial_1 \eta_1 \partial_2 \eta_1 \partial_3 \eta_1 \partial_3 \eta_2 \partial_2 \eta_3 \partial_3 \eta_3) \partial_1^2.$$

We have

$$\begin{aligned} \eta_1 \partial_1 \eta_1 \partial_1 \bullet R_{\eta_2 \eta_3; \partial_1^2} &= 0, \\ \eta_2 \partial_2 \eta_1 \partial_1 \bullet R_{\eta_1 \eta_3; \partial_1^2} &= 0, \\ \eta_3 \partial_3 \eta_1 \partial_1 \bullet R_{\eta_1 \eta_2; \partial_1^2} &= 0. \end{aligned}$$

Therefore $\eta_1 \eta_2 \eta_3 \partial_1^3$ -part of $F \bullet F^{\circ 2} \bullet F^{\circ 2}$ is equal to

$$\begin{aligned} \eta_1 \partial_1 \eta_1 \partial_1 \bullet R_{\eta_2 \eta_3; \partial_1^2} + \eta_2 \partial_2 \eta_1 \partial_1 \bullet R_{\eta_1 \eta_3; \partial_1^2} + \eta_3 \partial_3 \eta_1 \partial_1 \bullet R_{\eta_1 \eta_2; \partial_1^2} \\ = 0. \end{aligned}$$

By symmetry, $\eta_1 \eta_2 \eta_3 \partial_1^\alpha$ -part of $F \bullet F^{\circ 2} \bullet F^{\circ 2}$ are also 0 for any $\alpha \in \Gamma_3$, such that $|\alpha| = 3$. As we mentioned above η_i - and $\eta_i \eta_j$ -parts of elements obtained by two bullets are equal to 0. Lemma is proved.

By Lemma 5.9

$$\tau_3(F^5) = 5(2F^{\circ 3} \bullet F^{\bullet 2} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2}).$$

Therefore, we come to the following

Conclusion. $\tau_3(D^{10}) = 0$.

Proof of Theorem 0.1

By Theorem 0.1 of [2] if $\eta_{i_1} \cdots \eta_{i_r} \partial^\alpha$ -part of D^{10} is nonzero, then

$$|\alpha| \leq 3.$$

So,

$$D^{10} = \tau_1(D^{10}).$$

11. N -COMMUTATORS AND SUPER-DERIVATIONS

In this section we explain how escort invariants appear in calculating powers of odd derivations.

Suppose now $I = \{1, \dots, n\}$ and $D = \sum_{i=1}^n u_i \partial_i \in \text{Der } \mathcal{L}$ odd super-derivation. For $\alpha \in \mathbf{Z}_+^n$ set

$$x^{(\alpha)} = \frac{x^\alpha}{\alpha!}.$$

Denote by $\text{Supp}(s_k)$ a set of k -types $\{(\alpha^{(1)}, i_1), \dots, (\alpha^{(k)}, i_k)\}$ with $\alpha^{(1)}, \dots, \alpha^{(k)} \in \mathbf{Z}_+^n, i_1, \dots, i_k \in I$, such that $\sum_{p=1}^k \alpha^{(p)} - \epsilon_{i_p}$ has a form $-\beta$ for some $0 \neq \beta \in \mathbf{Z}_+^n$.

Theorem 11.1.

$$k! D^k =$$

$$\sum \partial^\alpha(u_{i_1}) \partial^\beta(u_{i_2}) \cdots \partial^\gamma(u_{i_k}) \text{esc}(s_k)(x^{(\alpha)} \partial_{i_1}, x^{(\beta)} \partial_{i_2}, \dots, x^{(\gamma)} \partial_{i_k}),$$

where summation is given by $\{(\alpha, i_1), (\beta, i_2), \dots, (\gamma, i_k)\} \in \text{Supp}(s_k)$.

If we use order on basic elements $x^\alpha \partial_i$ we can omit the coefficient $k!$:

$$D^k = \sum_{(\alpha^{(1)}, i_1) < \dots < (\alpha^{(k)}, i_k)} \partial^{\alpha^{(1)}}(u_{i_1}) \dots \partial^{\alpha^{(k)}}(u_{i_k}) \text{esc}(s_k)(x^{(\alpha^{(1)})} \partial_{i_1}, \dots, x^{(\alpha^{(k)})} \partial_{i_k}).$$

Proof. Recall that $U = \mathbf{C}[x_1, \dots, x_n]$ and ∂_i are partial derivations of U . Let Gr_k be a Grassman algebra with exterior generators η_1, \dots, η_k , i.e., it is associative super-commutative algebra of dimension 2^k . For $U = \mathbf{C}[x_1, \dots, x_n]$ take its Grassman envelope

$$\mathcal{U} = U \otimes Gr_k.$$

Prolong derivation $\partial_i \in \text{Der } U$ to a derivation of \mathcal{U} by

$$\partial_i(v \otimes \omega) = \partial_i(v) \otimes \omega.$$

We obtain commuting system of even derivations $\mathcal{D} = \{\partial_1, \dots, \partial_n\}$ of \mathcal{U} . For any $f_1, \dots, f_n \in \mathcal{U}$ and $\alpha \in \mathbf{Z}_+^n$ elements $\partial^\alpha u_j$ are odd. So, we obtain \mathcal{D} -differential super-algebra \mathcal{U} and we can consider its algebra of super-derivations $\mathcal{L} = \langle f \partial_i : f \in \mathcal{U} \rangle$ and its algebra of super-differential operators

$$\text{Diff} = \langle f \partial^\alpha : \alpha \in \mathbf{Z}_+^n, f \in \mathcal{U} \rangle.$$

We can endow Diff by composition operation, by left-symmetric multiplication and by bullet multiplication. In particular, we can consider \mathcal{L} as a left-symmetric algebra and as a super-Lie algebra. Thus,

$$\mathcal{L} \cong W_n \otimes Gr_k$$

is isomorphic to a current algebra with coefficients not in Laurent polynomials as usual, but in exterior algebra.

We see that for any f_1, \dots, f_n we can consider a homomorphism

$$\mathcal{L}_n \rightarrow \mathcal{U}, u_i \mapsto f_i, i = 1, \dots, n.$$

and this homomorphism can be extended to a homomorphism of left-symmetric or Lie algebras

$$\text{Der } \mathcal{L}_n \rightarrow \mathcal{L}$$

and to a homomorphism of associative (left-symmetric) algebras

$$\text{Diff} \rightarrow \text{Diff}$$

We can use this homomorphism in calculating F^k for $F = \sum_{i=1}^n f_i \partial_i \in \mathcal{L}$. In other words, in the formula for D^k we can make substitutions $u_i \mapsto f_i$ and calculate obtained expressions in \mathcal{U} .

Use this method for calculating coefficients $\lambda_{\{(\alpha, i_1), (\beta, i_2), \dots, (\gamma, i_k); \mu\}}$, where

$$D^k = \sum \lambda_{\{(\alpha, i_1), (\beta, i_2), \dots, (\gamma, i_k); \mu\}} \partial^\alpha(u_{i_1}) \partial^\beta(u_{i_2}) \dots \partial^\gamma(u_{i_k}) \partial^\mu.$$

Since numbers of u_i -indexes and ∂_i -indexes are equal, summation here is done by $\alpha, \beta, \dots, \gamma, \mu \in \mathbf{Z}_+^n$ and $i_1, \dots, i_k \in \{1, \dots, n\}$ such that $\alpha + \beta + \dots + \gamma + \mu = \sum_{s=1}^k \epsilon_{i_s}$. In other words, summation here is done by $\{(\alpha, i_1), (\beta, i_2), \dots, (\gamma, i_k)\} \in \text{Supp}_{|\mu|}(s_k)$.

Take

$$F = X_1 \otimes \eta_1 + \dots + X_k \otimes \eta_k \in \mathcal{L},$$

where $X_i \in W_n, i = 1, \dots, k$ are even elements. It is evident that

$$F^k = s_k(X_1, \dots, X_k) \otimes (\eta_1 \cdots \eta_k).$$

On the other hand, if $X_1 = x^{\alpha^{(1)}} \partial_{i_1}, X_2 = x^{\alpha^{(2)}} \partial_{i_2}, \dots, X_k = x^{\alpha^{(k)}} \partial_{i_k}$, then F can be presented in the form

$$F = \sum_{i=1}^k f_i \partial_i \in \mathcal{L},$$

where

$$f_i = \sum_{s: i_s=i} x^{\alpha^{(s)}} \otimes \eta_s \in \mathcal{U}$$

summation by s such that $i_s = i$. So, substitutions

$$u_i \mapsto \sum_{s: i_s=i} x^{\alpha^{(s)}} \otimes \eta_s \in \mathcal{U}$$

in D^k and making calculations in \mathcal{U} gives us on the one side

$$\lambda_{\{\alpha^{(1)}, i_1\}, \dots, \{\alpha^{(k)}, i_k\}; \mu} \alpha^{(1)}! \cdots \alpha^{(k)}! \partial^\mu \otimes \eta_1 \cdots \eta_k + Y,$$

where

$$Y \in \langle x^\alpha \partial^\beta \otimes Gr_k : |\alpha| > 0, \alpha, \beta \in \mathbf{Z}_+^n \rangle,$$

and on the other side

$$s_k(x^{\alpha^{(1)}} \partial_{i_1}, \dots, x^{\alpha^{(k)}} \partial_{i_k}) \otimes \eta_1 \cdots \eta_k.$$

Take projections $\text{Diff} \rightarrow \langle 1 \rangle \otimes \eta_1 \cdots \eta_k$ from the both parts. We have

$$\lambda_{\{\alpha^{(1)}, i_1\}, \dots, \{\alpha^{(k)}, i_k\}; \mu} \alpha^{(1)}! \cdots \alpha^{(k)}! \partial^\mu = \text{esc}(s_k)(x^{\alpha^{(1)}} \partial_{i_1}, \dots, x^{\alpha^{(k)}} \partial_{i_k}).$$

Thus,

$$\text{esc}(s_k)(x^{\alpha^{(1)}} \partial_{i_1}, \dots, x^{\alpha^{(k)}} \partial_{i_k}) = \lambda_{\{\alpha^{(1)}, i_1\}, \dots, \{\alpha^{(k)}, i_k\}; \mu} \partial^\mu$$

that we need to prove. \square

Recall that k -commutator s_k is called well defined on W_n if

$$\forall X_1, \dots, X_k \in W_n \Rightarrow s_k(X_1, \dots, X_k) \in W_n$$

Denote by s_k° a map $\wedge^k W_n \rightarrow W_n$ given by

$$s_k^\circ(X_1, \dots, X_k) = \sum_{\sigma \in \text{Sym}_k} \text{sign } \sigma X_{\sigma(1)} \circ (X_{\sigma(2)} \circ (\cdots (X_{\sigma(k-1)} \circ X_{\sigma(k)}))),$$

where W_n is considered as a left-symmetric algebra under multiplication $f\partial_i \circ g\partial_j = f\partial_i(g)\partial_j$.

Corollary 11.2. *The following conditions are equivalent:*

- $D^k \in \text{Der } \mathcal{L}$
- $D^k = D^{\circ k}$
- s_k is well defined operation on W_n .
- $s_k = s_k^\circ$.

Theorem 11.1 has two-fold applications. We use it in constructing D^k by s_k and vice versa one can use D^k in calculating k -commutators.

Example 1. Let $k = 1$. Then $s_1(x^\alpha \partial_i) = x^\alpha \partial_i$, therefore $\text{supp}(s_1) = \langle \partial_i : i = 1, \dots, n \rangle$ and $\text{Supp}(s_1) = \{(0, 1), \dots, (0, n)\}$. Hence

$$D^1 = \sum_{i=1}^n \partial^0(u_i) s_1(\partial_i) = \sum_{i=1}^n u_i \partial_i,$$

that we know well.

Example 2. Let us calculate $Y = s_{11}(X_1, \dots, X_{11})$ for $X_i = \partial_i, 1 \leq i \leq 3$, $X_4 = x_1 \partial_1 - x_3 \partial_3$, $X_5 = x_2 \partial_1$, $X_6 = x_3 \partial_1$, $X_7 = x_1 \partial_2$, $X_8 = x_2 \partial_2 - x_3 \partial_3$, $X_9 = x_1 \partial_3$, $X_{10} = x_2 \partial_3$, $X_{11} = x_3^2 \partial_1$. Let Gr_{11} be Grassman algebra generated by 11 odd elements η_1, \dots, η_{11} . Take $\mathcal{U} = \mathbf{C}[x_1, x_2, x_3] \otimes Gr_{11}$. Recall that x_i and ∂_i are even variables. Consider a homomorphism of super-differential polynomials algebra \mathcal{L}_3 to \mathcal{U} given by

$$u_1 \mapsto \eta_1 + x_1 \eta_4 + x_2 \eta_5 + x_3 \eta_6 + x_3^2 \eta_{11},$$

$$u_2 \mapsto \eta_2 + x_1 \eta_7 + x_2 \eta_8,$$

$$u_3 \mapsto \eta_3 - x_3 \eta_4 - x_3 \eta_8 + x_1 \eta_9 + x_2 \eta_{10}.$$

In other words make in the formulas for $\tau_1(D^{11})$ and $\tau_2(D^{11})$ corresponding substitutions. Make all calculations in \mathcal{L}_3 using the formula $\partial_i(v \otimes \eta) = \partial_i(v) \otimes \eta$, $v \in \mathbf{C}[x_1, x_2, x_3], \eta \in Gr_{11}$. One obtains that the linear part of Y is equal to 0 and the quadratic part of Y is equal to $80\partial_1^2$. So, $Y = 80\partial_1^2$.

The following results about N -commutators on $\text{Vect}(2)$ and $\text{Vect}_0(2)$ was established in [1]. 6-commutator on W_2 is well defined and it has one escort invariant

$$\text{escort}_{231} : L_0 \otimes L_1 \rightarrow R(\pi_1) \cong L_{-1},$$

$$\text{escort}_{231}(a, X) = d(a \circ \text{Div } X) + \text{Div } a \, d \, \text{Div } X - 3d \, \text{Div}(a \circ X).$$

$s_6 = 0$ is identity on S_1 and 5-commutator is well defined on S_1 and it has one escort invariant

$$\text{escort}_{221} : R(2\pi_1) \otimes R(3\pi_1) \rightarrow R(\pi_1) \cong L_{-1},$$

$$escort_{221}(a, X) = -3d \operatorname{Div}(a \circ X).$$

Below we do similar things for $Vect(3)$ and $Vect_0(3)$. Since calculations are too tedious and similar to calculations given above we formulate final results and give some examples.

12. 13-COMMUTATOR ON $Vect(3)$

Theorem 12.1. *If $n = 3$, then 13-commutator on W_3*

$$X_1, \dots, X_{13} \in W_3 \Rightarrow s_{13}(X_1, \dots, X_{13}) \in W_3.$$

It has one escort invariant

$$escort_{382} = escort(s_{13}) : L_0 \otimes \wedge^2 L_1 \rightarrow \wedge^2 R(\pi_1) \cong L_{-1}$$

defined by

$$\begin{aligned} escort_{382}(a, X, Y) = & \\ & -d(a \circ \operatorname{Div} X) \wedge d(\operatorname{Div} Y) + d(a \circ \operatorname{Div} Y) \wedge d(\operatorname{Div} X) \\ & -2(\operatorname{Div} a)d(\operatorname{Div} X) \wedge d(\operatorname{Div} Y) \\ & +4d(\operatorname{Div} X \circ a) \wedge d(\operatorname{Div} Y) + 4d(\operatorname{Div} X) \wedge d(\operatorname{Div} Y \circ a) \\ & +8(da \overset{\circ}{\wedge} dX) \circ \operatorname{Div} Y - 8(da \overset{\circ}{\wedge} dY) \circ \operatorname{Div} X. \end{aligned}$$

Corollary 12.2. *$escort_{382}$ induces a homomorphism $\bar{L}_0 \otimes \bar{L}_1 \otimes R(2\pi_1) \rightarrow \wedge^3 R(\pi_1)$ by*

$$(a\partial_i, v\partial_j, w) \mapsto da \wedge d\partial_i(v) \wedge d\partial_j(w).$$

Corollary 12.3. *$s_{13} = 0$ is identity on $Vect_0(3)$. Moreover $s_{12} = 0$ is identity on $Vect_0(3)$.*

Let

$$\begin{aligned} G_{ij}(a) &= \partial_i(a)\partial_j - \partial_j(a)\partial_i, \\ \tilde{u} &= u(x_1\partial_1 + x_2\partial_2 + x_3\partial_3). \end{aligned}$$

Take place isomorphisms of sl_n -modules

$$L_1 = \bar{L}_1 + \tilde{L}_1,$$

where

$$\bar{L}_1 = \langle X : \operatorname{Div}(X) = 0 \rangle \cong R(2\pi_1 + \pi_{n-1}) \cong R(2\pi_1 + \pi_{n-1}),$$

$$\tilde{L}_1 \cong R(\pi_1).$$

Then \bar{L}_1 is generated by elements of the form $G_{ij}(x^\alpha)$, where $i < j$, $\alpha \in \mathbf{Z}_+^n, |\alpha| = 3$ and \tilde{L}_1 has a basis $\{\tilde{x}_i : i = 1, \dots, n\}$.

Let us give construction of escort invariant in terms of \bar{L}_i and \tilde{L}_i . We see that $\text{escort}(s_{13})(a, X, Y) = 0$, if $X, Y \in \tilde{L}_1$ or $X, Y \in \bar{L}_1$ or $a = \tilde{1} = x_1\partial_1 + x_2\partial_2 + x_3\partial_3$. Below we use the following notation

$$a^{(i,j,k)} = \partial_1^i \partial_2^j \partial_3^k(a).$$

Non-zero components of $\text{escort}(s_{13})(a, X, Y)$ can be given by:

$$\text{escort}(s_{13})(G_{12}(a), G_{12}(b), \tilde{x}_1) =$$

$$\begin{aligned} & (-32a^{(1,0,1)}b^{(0,3,0)} + 32a^{(1,1,0)}b^{(0,2,1)} - 32a^{(0,2,0)}b^{(1,1,1)} + 32a^{(0,1,1)}b^{(1,2,0)})\partial_1 \\ & + (32a^{(1,0,1)}b^{(1,2,0)} - 16a^{(2,0,0)}b^{(0,2,1)} + 16a^{(0,2,0)}b^{(2,0,1)} - 32a^{(0,1,1)}b^{(2,1,0)})\partial_2 \\ & + (-32a^{(1,1,0)}b^{(1,2,0)} + 16a^{(2,0,0)}b^{(0,3,0)} + 16a^{(0,2,0)}b^{(2,1,0)})\partial_3, \end{aligned}$$

$$\text{escort}(s_{13})(G_{12}(a), G_{12}(b), \tilde{x}_2) =$$

$$\begin{aligned} & (32a^{(1,0,1)}b^{(1,2,0)} - 16a^{(2,0,0)}b^{(0,2,1)} + 16a^{(0,2,0)}b^{(2,0,1)} - 32a^{(0,1,1)}b^{(2,1,0)})\partial_1 \\ & + (32a^{(2,0,0)}b^{(1,1,1)} - 32a^{(1,1,0)}b^{(2,0,1)} - 32a^{(1,0,1)}b^{(2,1,0)} + 32a^{(0,1,1)}b^{(3,0,0)})\partial_y \\ & + (-16a^{(2,0,0)}b^{(1,2,0)} + 32a^{(1,1,0)}b^{(2,1,0)} - 16a^{(0,2,0)}b^{(3,0,0)})\partial_3 \end{aligned}$$

$$\text{escort}(s_{13})(G_{12}(a), G_{12}(b), \tilde{x}_3) =$$

$$\begin{aligned} & (-32a^{(1,1,0)}b^{(1,2,0)} + 16a^{(2,0,0)}b^{(0,3,0)} + 16a^{(0,2,0)}b^{(2,1,0)})\partial_1 \\ & + (-16a^{(2,0,0)}b^{(1,2,0)} + 32a^{(1,1,0)}b^{(2,1,0)} - 16a^{(0,2,0)}b^{(3,0,0)})\partial_2 \end{aligned}$$

$$\text{escort}(s_{13})(G_{12}(a), G_{23}(b), \tilde{x}_1) =$$

$$\begin{aligned} & (-16a^{(1,0,1)}b^{(0,2,1)} - 16a^{(0,2,0)}b^{(1,0,2)} + 16a^{(1,1,0)}b^{(0,1,2)} + 16a^{(0,1,1)}b^{(1,1,1)})\partial_2 \\ & + (+16a^{(0,2,0)}b^{(1,1,1)} - 16a^{(0,1,1)}b^{(1,2,0)} - 16a^{(1,1,0)}b^{(0,2,1)} + 16a^{(1,0,1)}b^{(0,3,0)})\partial_3 \end{aligned}$$

$$\text{escort}(s_{13})(G_{12}(a), G_{23}(b), \tilde{x}_2) =$$

$$\begin{aligned} & (-16a^{(1,0,1)}b^{(0,2,1)} - 16a^{(0,2,0)}b^{(1,0,2)} + 16a^{(1,1,0)}b^{(0,1,2)} + 16a^{(0,1,1)}b^{(1,1,1)})\partial_1 \\ & + (32a^{(1,1,0)}b^{(1,0,2)} + 32a^{(1,0,1)}b^{(1,1,1)} - 32a^{(2,0,0)}b^{(0,1,2)} - 32a^{(0,1,1)}b^{(2,0,1)})\partial_2 \\ & + (-48a^{(1,1,0)}b^{(1,1,1)} - 16a^{(1,0,1)}b^{(1,2,0)} + 32a^{(2,0,0)}b^{(0,2,1)} \\ & + 16a^{(0,2,0)}b^{(2,0,1)} + 16a^{(0,1,1)}b^{(2,1,0)})\partial_3 \end{aligned}$$

$$\begin{aligned}
& escort(s_{13})(G_{12}(a), G_{23}(b), \tilde{x}_3) = \\
& (16a^{(1,0,1)}b^{(0,3,0)} - 16a^{(1,1,0)}b^{(0,2,1)} + 16a^{(0,2,0)}b^{(1,1,1)} - 16a^{(0,1,1)}b^{(1,2,0)})\partial_1 \\
& + (-48a^{(1,1,0)}b^{(1,1,1)} - 16a^{(1,0,1)}b^{(1,2,0)} + 32a^{(2,0,0)}b^{(0,2,1)} \\
& + 16a^{(0,2,0)}b^{(2,0,1)} + 16a^{(0,1,1)}b^{(2,1,0)})\partial_2 \\
& + (-32a^{(2,0,0)}b^{(0,3,0)} + 64a^{(1,1,0)}b^{(1,2,0)} - 32a^{(0,2,0)}b^{(2,1,0)})\partial_3.
\end{aligned}$$

If $\{i, j, s, k\} \subseteq \{1, 2, 3\}$, then any two pairs $(i, j), (s, k)$ has at least one common element. Therefore by symmetry one can easily write other formulas for $escort(s_{13})(G_{ij}(a), G_{sk}(b), \tilde{x}_r)$.

Let us show how to use theorem 12.1 in calculation of 13-commutator on $Vect(3)$.

Example 1. Take

$$\begin{aligned}
X_1 &= \partial_1, X_2 = \partial_2, X_3 = \partial_3, X_4 = x_1\partial_1, X_5 = x_2\partial_1, X_6 = x_3\partial_1, X_7 = x_1\partial_2, \\
X_8 &= x_2\partial_2, X_9 = x_3\partial_2, X_{10} = x_1\partial_3, X_{11} = x_2\partial_3, X_{12} = x_1x_2\partial_3, X_{13} = x_3^2\partial_3.
\end{aligned}$$

We see that the number of elements of grade -1 is 3 and the number elements of grade 0 is 8. In 0-part here appear all base elements of gl_3 except $a = x_3\partial_3$. Elements of the grade 1 are two: $X = X_{12}$ and $Y = X_{13}$. So, to calculate 13-commutator of 13 elements X_1, \dots, X_{13} , we denote it $s_{13}(X_1, \dots, X_{13})$, we need to calculate $escort(s_{13})(a, X, Y)$. We have

$$Div X = \partial_3(x_1x_2) = 0,$$

therefore,

$$-d(a \circ Div X) \wedge d(Div Y) + d(a \circ Div Y) \wedge d(Div X) - 2(Div a) d(Div X) \wedge d(Div Y) = 0.$$

Further,

$$Div a = \partial_3(x_3) = 1, \quad Div X \circ a = Div(x_1x_2\partial_3(x_3)\partial_3) = 0,$$

and

$$+4 d(Div X \circ a) \wedge d(Div Y) + 4 d(Div X) \wedge d(Div Y \circ a) = 0.$$

Finally, $Div Y = 2x_3$ and

$$\begin{aligned}
& 8 \left\{ \sum_{i,j=1}^3 \partial_i(Div Y) d(a(x_j)) \wedge d(\partial_j(X(x_i))) - \partial_i(Div X) d(a(x_j)) \wedge d(\partial_j(Y(x_i))) \right\} = \\
& 8 \sum_{i,j=1}^3 \partial_i(2x_3) d(a(x_j)) \wedge d(\partial_j(x_1x_2\partial_3(x_i))) =
\end{aligned}$$

$$16 \sum_{j=1}^3 d(a(x_j)) \wedge d(\partial_j(x_1 x_2)) = 0.$$

Therefore,

$$escort(s_{13})(x_3 \partial_3, x_1 x_2 \partial_3, x_3^2 \partial_3) = 0.$$

Example 2. Now change in example 1 X_{12} to

$$X_{12} = x_1 x_3 \partial_3,$$

other elements are as before. Two vector fields, nameley $X = X_{12}$ and $Y = X_{13}$ have non-constant divergences:

$$Div X = x_1, Div Y = 2x_3.$$

So, we can expect that $s_{11}(X_1, \dots, X_{13})$ might be non-trivial vector field. We have

$$\begin{aligned} -d(a \circ Div X) \wedge d(Div Y) + d(a \circ Div Y) \wedge d(Div X) = \\ 2dx_3 \wedge dx_1, \end{aligned}$$

$$-2(Div a)d(Div X) \wedge d(Div Y) = -4dx_1 \wedge dx_3,$$

$$\begin{aligned} 4d(Div X \circ a) \wedge d(Div Y) + 4d(Div X) \wedge d(Div Y \circ a) = \\ 8dx_1 \wedge dx_3 + 8dx_1 \wedge dx_3, \end{aligned}$$

$$\begin{aligned} 8 \left\{ \sum_{i,j=1}^3 \partial_i(Div Y) d(a(x_j)) \wedge d(\partial_j(X(x_i))) - \partial_i(Div X) d(a(x_j)) \wedge d(\partial_j(Y(x_i))) \right\} = \\ 8 \{ 2dx_3 \wedge d(\partial_3(x_1 x_3)) = \\ -16 dx_1 \wedge dx_3. \end{aligned}$$

Thus,

$$escort(s_{13})(a, X, Y) = -6dx_1 \wedge dx_3.$$

Since the isomorphism $\wedge^2 R(\pi_1) \cong R(\pi_2)$ is established by

$$dx_1 \wedge dx_2 \mapsto \partial_3, dx_1 \wedge dx_3 \mapsto -\partial_2, dx_2 \wedge dx_3 \mapsto \partial_1$$

this means that

$$s_{13}(X_1, \dots, X_{13}) = esc(s_{13})(X_1, \dots, X_{13}) = 6\partial_2.$$

Example 3. Let now

$$X_{12} = x_1 x_3 \partial_3, X_{13} = x_2 x_3 \partial_3.$$

other elements as above. Then

$$-d(a \circ Div X) \wedge d(Div Y) + d(a \circ Div Y) \wedge d(Div X) = 0,$$

$$-2(\operatorname{Div} a)d(\operatorname{Div} X) \wedge d(\operatorname{Div} Y) = -2dx_1 \wedge dx_2,$$

$$4d(\operatorname{Div} X \circ a) \wedge d(\operatorname{Div} Y) + 4d(\operatorname{Div} X) \wedge d(\operatorname{Div} Y \circ a) = \\ 4dx_1 \wedge dx_2 + 4dx_1 \wedge dx_2 = 8dx_1 \wedge dx_2,$$

$$\sum_{i,j=1}^3 \partial_i(\operatorname{Div} Y)d(a(x_j)) \wedge d(\partial_j(X(x_i))) - \partial_i(\operatorname{Div} X)d(a(x_j)) \wedge d(\partial_j(Y(x_i))) = 0.$$

Thus,

$$\operatorname{escort}(s_{13})(a, X, Y) = 6dx_1 \wedge dx_2,$$

and

$$s_{13}(X_1, \dots, X_{13}) = \operatorname{escort}(s_{13})(X_1, \dots, X_{13}) = 6\partial_3.$$

Example 4. Let now all $Y_i = X_i$ as before if $i < 12$ and

$$Y_{12} = x_1x_3\partial_3, Y_{13} = x_2x_3^2\partial_3.$$

Then

$$s_{13}(Y_1, \dots, Y_{13}) = E_{x_3^2\partial_3}(Y_{13})\operatorname{esc}(s_{13})(X_1, \dots, X_{11}, x_1x_3\partial_3, x_3^2\partial_3) + \\ E_{x_2x_3\partial_3}(Y_{13})\operatorname{esc}(s_{13})(X_1, \dots, X_{11}, x_1x_3\partial_3, x_2x_3\partial_3) = \\ \text{(by results of example 2 and 3)}$$

$$6x_2\partial_2 + 12x_3\partial_3.$$

13. 10-COMMUTATOR ON $\operatorname{Vect}(3)$ AND $\operatorname{Vect}_0(3)$

Recall some denotions:

$$U_k = \langle x^\alpha | |\alpha| = k \rangle,$$

the multiplication \circ is left-symmetric and $\overset{\circ}{\wedge}$ means wedge-product corresponding to left-symmetric multiplication:

$$L_1 \times U_2 \rightarrow \wedge^2 U_1, \quad (u\partial_i, v) \mapsto du\partial_i \overset{\circ}{\wedge} dv := du \wedge d\partial_i(v).$$

Below expressions like $\operatorname{Div} a \circ X$ will mean $\operatorname{Div}(a \circ X)$.

Theorem 13.1. 10-commutator is well defined on $L = \operatorname{Vect}(3)$:

$$X_1, \dots, X_{10} \in L \Rightarrow s_{10}(X_1, \dots, X_{10}) \in L.$$

It has three escort invariants

$$\operatorname{escort}_{271} : R(\pi_1) \otimes \wedge^2 L_0 \otimes L_2 \rightarrow \wedge^2 R(\pi_1) \cong L_{-1},$$

$$\operatorname{escort}_{3601} : \wedge^3 L_0 \otimes L_2 \rightarrow \wedge R(\pi_1) \cong L_{-1},$$

and

$$escort_{352} : \wedge^4 L_0 \otimes \wedge^2 L_1 \rightarrow \wedge^2 R(\pi_1) \cong L_{-1}.$$

They can be given by

$$escort_{271}(u, a, b, X) =$$

$$\begin{aligned} & du \wedge (11d(\text{Div}([a, b] \circ X)) + 21d(a \circ \text{Div}(b \circ X)) - b \circ \text{Div}(a \circ X)) - 44d([a, b] \circ \text{Div} X)) \\ & - 32(d(a \circ u) \wedge d(\text{Div}(b \circ X)) - d(b \circ u) \wedge d(\text{Div}(a \circ X))) \\ & - 50(d(a \circ u) \wedge d(b \circ \text{Div} X) - d(b \circ u) \wedge d(a \circ \text{Div} X)) \\ & + \text{Div} a \, du \wedge d(2b \circ \text{Div} X + 9\text{Div}(b \circ X)) - \text{Div} b \, du \wedge d(2a \circ \text{Div} X + 9\text{Div}(a \circ X)) \\ & + 8(\text{Div} a \, db \overset{\circ}{\wedge} d(u \text{Div} X) - \text{Div} b \, da \overset{\circ}{\wedge} d(u \text{Div} X)) \\ & + 12(da \overset{\circ}{\wedge} d((b \circ X) \circ u) - db \overset{\circ}{\wedge} d((a \circ X) \circ u)) \\ & - 28d[a, b] \overset{\circ}{\wedge} d(X \circ u) \\ & + 16(da \overset{\circ}{\wedge} d(X \circ (b \circ u)) - db \overset{\circ}{\wedge} d(X \circ (a \circ u))), \end{aligned}$$

$$escort_{3601}(a, b, c, X) =$$

$$\begin{aligned} & -6(da \overset{\circ}{\wedge} d(b \circ (\text{Div}(c \circ X))) + db \overset{\circ}{\wedge} d(c \circ (\text{Div}(a \circ X))) + dc \overset{\circ}{\wedge} d(a \circ (\text{Div}(b \circ X)))) \\ & + 6(da \overset{\circ}{\wedge} d(c \circ (\text{Div}(b \circ X))) + db \overset{\circ}{\wedge} d(a \circ (\text{Div}(c \circ X))) + dc \overset{\circ}{\wedge} d(b \circ (\text{Div}(a \circ X)))) \\ & - 5(da \overset{\circ}{\wedge} d(\text{Div}([b, c] \circ X)) + db \overset{\circ}{\wedge} d(\text{Div}([c, a] \circ X)) + dc \overset{\circ}{\wedge} d(\text{Div}([a, b] \circ X))) \\ & - d[a, b] \overset{\circ}{\wedge} d(\text{Div}(c \circ X)) - d[b, c] \overset{\circ}{\wedge} d(\text{Div}(a \circ X)) - d[c, a] \overset{\circ}{\wedge} d(\text{Div}(b \circ X)) \\ & - 12(d[a, b] \overset{\circ}{\wedge} d(c \circ \text{Div} X) + d[b, c] \overset{\circ}{\wedge} d(a \circ \text{Div} X) + d[c, a] \overset{\circ}{\wedge} d(b \circ \text{Div} X)) \\ & + 27d(a \circ [b, c] + b \circ [c, a] + c \circ [a, b]) \overset{\circ}{\wedge} d(\text{Div} X) \\ & + 18\text{Div} a \, (db \overset{\circ}{\wedge} d(c \circ \text{Div} X) - dc \overset{\circ}{\wedge} d(b \circ \text{Div} X)) \\ & + 18\text{Div} b \, (dc \overset{\circ}{\wedge} d(a \circ \text{Div} X) - db \overset{\circ}{\wedge} d(a \circ \text{Div} X)) \\ & + 18\text{Div} c \, (da \overset{\circ}{\wedge} d(b \circ \text{Div} X) - db \overset{\circ}{\wedge} d(a \circ \text{Div} X)) \\ & - 14\text{Div} a \, (db \overset{\circ}{\wedge} d(\text{Div}(c \circ X)) - dc \overset{\circ}{\wedge} d(\text{Div}(b \circ X))) \\ & - 14\text{Div} b \, (dc \overset{\circ}{\wedge} d(\text{Div}(a \circ X)) - da \overset{\circ}{\wedge} d(\text{Div}(c \circ X))) \\ & - 14\text{Div} c \, (da \overset{\circ}{\wedge} d(\text{Div}(b \circ X)) - db \overset{\circ}{\wedge} d(\text{Div}(a \circ X))) \end{aligned}$$

We are not able to write escort invariant of type $(3, 5, 2)$ in a compact form.

Corollary 13.2. *10-commutator on $L = Vect_0(3)$ is well defined and escort invariants are given by*

$$escort_{271} : R(\pi_1) \otimes \wedge^2 R(\pi_1 + \pi_2) \otimes R(2\pi_1 + \pi_2) \rightarrow \wedge^2 R(\pi_1),$$

$$escort_{271}(u, a, b, X) =$$

$$\begin{aligned} & du \wedge (11d(Div([a, b] \circ X)) + 21d(a \circ Div(b \circ X) - b \circ Div(a \circ X))) \\ & - 32(d(a \circ u) \wedge d(Div(b \circ X)) - d(b \circ u) \wedge d(Div(a \circ X))) \\ & + 12(da \overset{\circ}{\wedge} d((b \circ X) \circ u) - db \overset{\circ}{\wedge} d((a \circ X) \circ u)) \\ & - 28d[a, b] \overset{\circ}{\wedge} d(X \circ u) \\ & + 16(da \overset{\circ}{\wedge} d(X \circ (b \circ u)) - db \overset{\circ}{\wedge} d(X \circ (a \circ u))), \end{aligned}$$

$$escort_{3601} : \wedge^3 R(\pi_1 + \pi_2) \otimes R(3\pi_1 + \pi_{n-1}) \rightarrow \wedge^2 R(\pi_1)$$

$$escort_{3601}(a, b, c, X) =$$

$$\begin{aligned} & -6(da \overset{\circ}{\wedge} d(b \circ (Div(c \circ X))) + db \overset{\circ}{\wedge} d(c \circ (Div(a \circ X))) + dc \overset{\circ}{\wedge} d(a \circ (Div(b \circ X)))) \\ & + 6(da \overset{\circ}{\wedge} d(c \circ (Div(b \circ X))) + db \overset{\circ}{\wedge} d(a \circ (Div(c \circ X))) + dc \overset{\circ}{\wedge} d(b \circ (Div(a \circ X)))) \\ & - 5(da \overset{\circ}{\wedge} d(Div([b, c] \circ X)) + db \overset{\circ}{\wedge} d(Div([c, a] \circ X)) + dc \overset{\circ}{\wedge} d(Div([a, b] \circ X))) \end{aligned}$$

$$escort_{352} : \wedge^4 R(\pi_1 + \pi_2) \otimes \wedge^2 R(2\pi_1 + \pi_2) \rightarrow \wedge^2 R(\pi_1) \cong L_{-1}$$

is too big to be presented here.

All coefficients are calculated by Mathematica and Maple. I am gratefull to Y.I. Manin and M. Jibladze for interest to 10- and 13-commutators.

REFERENCES

- [1] A.S. Dzhumadil'daev, *N-commutators*, Comment. Math. Helv. **79** (2004), no. 3, 516–553.
- [2] A.S. Dzhumadil'daev, *Odd derivations and N-commutators*, submitted.
- [3] A.S. Dzhumadil'daev, *A remark on the space of invariant differential operators*, Vestnik Moskov. Univ, Ser.1 Mat.Mekh, 1982, No.2, 49–54, 116=Engl. transl. Moscow Univ. Math. Bull., **37**(1982), No.2, p.63–68.

KAZAKH-BRITISH UNIVERSITY, TOLE BI, 59, ALMATY, 480091, KAZAKHSTAN
E-mail address: askar@math.kz